SUPERCONVERGENCE OF NEW MIXED FINITE ELEMENT SPACES

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Abstract. In this paper we prove some superconvergence of a new family of mixed finite element spaces of higher order we introduced in [ETNA, Vol.37, pp.189–201, 2010]. Among all the mixed finite element spaces having optimal order of convergence on quadrilateral grids, this space has the smallest unknowns. However, the scalar variable is only suboptimal in general; thus we have employed a post-processing technique for the scalar variable. As a byproduct, we have obtained a superconvergence on rectangular grid. The superconvergence of velocity variable naturally holds and can be shown by a minor modification of existing theory, but that of a scalar variable requires a new technique, especially for $k = 1$.Numerical experiments are provided to support the theory.

Key words. Mixed finite element method, superconvergence, optimal order, post-processing.

AMS subject classifications. 65N15, 65N30, 35J60

1. Introduction. Mixed finite element methods (MFEM) for elliptic problems [20, 18, 10, 3, 4, 5, 1, 6, 7, 8] have been developed for a more accurate approximation to some physical quantities, such as velocity of a fluid in the porous media equations or the stress in the elasticity equations, etc. The derivation of mixed finite element method is based on the introduction of a new variable obeying physical law, for instance, Darcy's law between pressure and velocity variables in porus media problems. Although introducing a new variable in MFEM requires an additional equation which represents the physical law, it allows us to compute the new physical variable and the primal variable simultaneously.

During the study of mixed finite element methods, one often observes an unexpectedly higher order convergence than the optimal order, called a superconvergence. There are quite a few results in this direction [11, 13, 12, 14, 15]. For example, a superconvergence between the finite element solution and the projection of exact solution is proved in [14, 11, 13]. The error between the finite element solution and the exact solution are usually measured in discrete norm along Gauss lines and shown in [11, 13, 12, 15].

Recently we have introduced a new family of mixed finite elements on quadrilateral grids in [19], lying between Raviart-Thomas (RT) [20] and Brezzi-Douglas-Fortin-Marini (BDFM) spaces. This new element shows optimal order $O(h^{k+1})$ in the velocity variable and suboptimal order $O(h^k)$ for pressure variable when essentially quadrilateral grid is used(of course the pressure variable is also optimal if rectangular grid is used). Thus we have used a post-processing for pressure to obtain optimal order for pressure in case of quadrilaterals. While there are many results on the superconvergence of vector variable, the superconvergence of pressure variable are rare, see [15, 12].

In this paper we shall prove the superconvergence of our mixed finite elements. Superconvergence for velocity variable easily follows as our space contains BDFM

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space. However, for pressure variable, we need a new technique to prove the superconvergence. When $k = 1$, our result shows one order higher than [21].

The organization of this paper is as follows: In the next section, we present basic materials necessary to study MFEM, and briefly explain our elements. In section 3, we present the post-processing scheme and the superconvergence. Finally, numerical results are given in section 4 which support our theory.

2. Mixed finite element. Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. We consider the following second-order elliptic boundary value problem:

$$\begin{aligned} -\mathrm{div}\,(\kappa\,\nabla p) &= f, \quad \text{in } \Omega, \\ p &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\kappa = \kappa(\mathbf{x})$ is a symmetric and uniformly positive definite matrix. Let us introduce a vector variable $\mathbf{u} = -\kappa\nabla p$ and reformulate the problem (2.1) in the mixed form

$$\begin{aligned} \mathbf{u} + \kappa\nabla p &= 0, \quad \text{in } \Omega, \\ \mathrm{div}\,\mathbf{u} &= f, \quad \text{in } \Omega, \\ p &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

For any domain Ω, we let $L^2(\Omega)$ be the set of all square integrable functions on Ω equipped with the usual inner product $(\cdot,\cdot)_\Omega$. Let $H^i(\Omega) = W^{i,2}(\Omega)$ be the Sobolev spaces of order $i = 0, 1, \cdots$, with obvious norms and let $W = L^2(\Omega)$. Now, let $\mathbf{H}^i(\Omega)$ be the space of vectors $\mathbf{u} = (u, v)$ each of whose component lies in $H^i(\Omega), i = 0, 1, \cdots$. For both of the spaces $H^i(\Omega)$ and $\mathbf{H}^i(\Omega), i = 0, 1, 2, \cdots$, we shall denote the norms(semi-norms) by $\|\cdot\|_{i,\Omega}(|\cdot|_{i,\Omega})$, or $\|\cdot\|_i(|\cdot|_i)$ when no confusion arises. Also, let $\mathbf{V} = H(\mathrm{div}\,;\Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 : \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}$ with norm $\|\mathbf{v}\|^2_{H(\mathrm{div}\,;\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)} + \|\mathrm{div}\,\mathbf{v}\|^2_{L^2(\Omega)}$. Then we have the following variational form for (2.2):

$$\begin{aligned} (\kappa^{-1}\mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\,\mathbf{v}) &= 0, \quad \forall \mathbf{v} \in \mathbf{V}, \\ (\mathrm{div}\,\mathbf{u}, q) &= (f, q), \quad \forall q \in W. \end{aligned} \tag{2.3}$$

This problem has a unique solution pair $(\mathbf{u}, p) \in \mathbf{V} \times W$. Now we consider finite element methods. From now on, we assume the domain can be partitioned into rectangles. For each $h > 0$, let $\mathscr{T}_h = \{K\}$ be a partition of the domain $\bar{\Omega}$ into closed rectangles whose side is bounded by h. Assume that we have some finite dimensional spaces $\mathbf{V}_h \subset \mathbf{V}$ and $W_h \subset W$ based on these grids. Then the corresponding finite dimensional problem becomes: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$\begin{aligned} (\kappa^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \mathrm{div}\,\mathbf{v}_h) &= 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\mathrm{div}\,\mathbf{u}_h, q_h) &= (f, q_h), \quad \forall q_h \in W_h. \end{aligned} \tag{2.4}$$

Let $\hat{K} = [-1, 1] \times [-1, 1]$ be the reference element. We define $\mathbf{V}_h(K)$ and $W_h(K)$ as follows:

$$\mathbf{V}_h(K) = \{\mathbf{v} = \mathcal{P}_K\hat{\mathbf{v}} : \hat{\mathbf{v}} \in \hat{\mathbf{V}}(\hat{K})\}, \tag{2.5}$$

$$W_h(K) = \{q = \hat{q} \circ F_K^{-1} : \hat{q} \in \hat{W}(\hat{K})\}, \tag{2.6}$$

where $F_K : \hat{K} \to K$ is an affine map and $\mathcal{P}_K : \mathbf{H}(\mathrm{div}\,; \hat{K}) \to \mathbf{H}(\mathrm{div}\,; K)$ is the Piola transform defined by

$$\mathbf{v} = \mathcal{P}_K \hat{\mathbf{v}} = \frac{DF_K}{J_K} \hat{\mathbf{v}} \circ F_K^{-1}.$$

Finally we define

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{V} : \mathbf{v}|_K \in \mathbf{V}_h(K)\}, \tag{2.7}$$
$$W_h = \{q \in L^2(\Omega) : q|_K \in W_h(K)\}. \tag{2.8}$$

The most common example is $RT_{[k]}$ given by Raviart and Thomas [20], where

$$\hat{\mathbf{V}}(\hat{K}) = Q_{k+1,k}(\hat{K}) \times Q_{k,k+1}(\hat{K}), \quad \hat{W}(\hat{K}) = Q_{k,k}(\hat{K}).$$

Here, $Q_{i,j}(\Omega)$ for any domain Ω is the space of polynomials of degree i and j in each variable. For later use, we shall denote by $P_k(\Omega)$ the space of polynomials of total degree k on Ω. For detailed descriptions of MFEM and other spaces like Brezzi-Douglas-Marini (BDM), BDFM spaces [3, 4]. For these spaces, the following results are well-known:

Theorem 2.1. *Let $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$ and $p \in H^{k+1}(\Omega)$ be the solution of (2.3) and $\mathbf{u}_h$ and p_h be the solution of (2.4). Then*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + \|p - p_h\|_0 + \|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_0 \le Ch^{k+1}(\|\mathbf{u}\|_{k+1} + \|\mathrm{div}\,\mathbf{u}\|_{k+1}). \tag{2.9}$$

2.1. A new mixed finite element spaces. In this subsection, we briefly describe the new mixed finite element spaces that we developed in [19]. First let $\mathcal{S}_k$ be a subspace of $RT_{[k]}$ space of order k, where the two elements $(\hat{x}^{k+1}\hat{y}^k, 0)^T$, $(0, \hat{x}^k\hat{y}^{k+1})^T$ are replaced by the single element $(\hat{x}^{k+1}\hat{y}^k, -\hat{x}^k\hat{y}^{k+1})^T$. Let $\mathcal{R}_k$ be the space of all polynomials in each variable up to degree $k+1$ except constant multiple of the term $\hat{x}^{k+1}\hat{y}^{k+1}$. Then the mixed finite element we study for quadrilateral grids is based on the pair $(\mathcal{S}_k, \mathcal{R}_{k-1})$ for $k \ge 1$. Define

$$\hat{\mathbf{V}}(\hat{K}) = \mathcal{S}_k, \ \hat{W}(\hat{K}) = \mathcal{R}_{k-1} \tag{2.10}$$

as reference spaces for our element and define $\mathbf{V}_h$ and W_h through (2.7) and (2.8). For the unisolvence of this element, let $\mathbf{\Psi}_k(\hat{K})$ be an subspace of $Q_{k-1,k} \times Q_{k,k-1}(\hat{K})$ where $(\hat{x}^{k-1}\hat{y}^k, 0)$ and $(0, \hat{x}^k\hat{y}^{k-1})$ are replaced by the single element $(\hat{x}^{k-1}\hat{y}^k, -\hat{x}^k\hat{y}^{k-1})$. Then we have the following lemma [19] :

Lemma 2.2 (Unisolvence). *For any $\hat{\mathbf{u}} = (\hat{u}, \hat{v}) \in \mathcal{S}_k$, the conditions*

$$\int_{\hat{e}} \hat{\mathbf{u}} \cdot \hat{\mathbf{n}}\, \hat{q}\, d\hat{s}, \quad \hat{q} \in P_k(\hat{e}), \quad \textit{for each edge } \hat{e} \textit{ of } \hat{K}, \tag{2.11}$$
$$\int_{\hat{K}} \hat{\mathbf{u}} \cdot \hat{\mathbf{v}}\, d\hat{\mathbf{x}}, \quad \hat{\mathbf{v}} \in \mathbf{\Psi}_k(\hat{K}) \tag{2.12}$$

uniquely determine $\hat{\mathbf{u}}$.

These conditions also determines the projection $\Pi_h : \mathbf{H}^{k+1}(\Omega) \to \mathbf{V}_h$. For the mixed finite space $(\mathcal{S}_k,\mathcal{R}_{k-1})$ the following estimates are proved in [19]:

THEOREM 2.3. *Let* $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$ *and* $p \in H^{k+1}(\Omega)$ *be the solution of (2.3) and* $\mathbf{u}_h$ *and* p_h *be the solution of (2.4). Then*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^{k+1}\|\mathbf{u}\|_{k+1}, \tag{2.13}$$

$$\|p - p_h\|_0 \le Ch^{j}\|p\|_{k+1} \tag{2.14}$$

where $j = k+1$ *for rectangle grid and* $j = k$ *for quadrilaterals.*

These estimates show an optimal order of convergence for the vector variable $\mathbf{u}$, but a loss of order for the scalar variables p for quadrilaterals. To obtain the optimal order for p, we employ a local post-processing scheme. In the next section we shall discuss the local post-processing scheme and prove its superconvergence.

3. Superconvergence. Basically there are two kinds of superconvergence results. One is the superconvergence between certain projection and finite element solution. This is usually measured in L^2-norm. Another type of superconvergence is between the true solution and finite element solution. This is usually measured in some discrete L^2-(semi) norm such as measured along Gauss lines. Most analyses are for vector variables and relatively few articles deal with the scalar variable.

We need two kinds of discrete inner products: one for scalar and the other for vector variable. For an element $K = [a,b] \times [c,d]$, let $x_1, x_2, \cdots, x_k$ and $y_1, y_2, \cdots, y_k$ be the Gaussian points in $[a,b]$ and $[c,d]$, respectively. Define discrete inner product for $H^{1+\epsilon}(K)(\epsilon > 0)$ by

$$\langle\!\langle u, v\rangle\!\rangle_{g_k} = \sum_{i,j=1}^{k} w_i w_j (b-a)(d-c) u(x_i, y_j) v(x_i, y_j), \tag{3.1}$$

$$\langle\!\langle u, v\rangle\!\rangle_{g_k}^{(1)} = \sum_{i=1}^{k} w_i (d-c) \int_a^b u(x, y_i) v(x, y_i)\, dx, \tag{3.2}$$

$$\langle\!\langle u, v\rangle\!\rangle_{g_k}^{(2)} = \sum_{i=1}^{k} w_i (b-a) \int_c^d u(x_i, y) v(x_i, y)\, dy, \tag{3.3}$$

where $w_i, i = 1, \cdots, k$ are the weights of Gaussian quadrature rules. The corresponding norms are:

$$\|u\|_{g_k}^2 = \sum_{i,j=1}^{k} w_i w_j (b-a)(d-c) |u(x_i, y_j)|^2 \tag{3.4}$$

$$\|u\|_{g_k^{(1)}}^2 = \sum_{i=1}^{k} w_i (d-c) \int_a^b |u(x, y_i)|^2\, dx, \tag{3.5}$$

$$\|v\|_{g_k^{(2)}}^2 = \sum_{i=1}^{k} w_i (b-a) \int_c^d |v(x_i, y)|^2\, dy. \tag{3.6}$$

For the vector variable $\mathbf{u} = (u, v) \in (H^{1+\epsilon}(K))^2$, we define

$$\|\mathbf{u}\|_{g_k}^2 = \|u\|_{g_k^{(1)}}^2 + \|v\|_{g_k^{(2)}}^2. \tag{3.7}$$

These notations naturally extend for functions defined on Ω.

We state known superconvergent results in mixed finite elements. First we consider the vector variable. The following results are known in [11, 13, 14, 16]. In fact, they were originally shown for $RT_{[k]}$, but later generalized to $BDFM_{[k+1]}$ element in [11]. Since our space contains $BDFM_{[k+1]}$, they hold for our spaces with minor modification.

THEOREM 3.1. *If $p \in H^{k+2}(\Omega)$ and $\mathbf{u} \in \mathbf{H}^{k+2}(\Omega)$ are the solutions of (2.2) and if $(\mathbf{u}_h, p_h)$ are the solutions of (2.4) with $RT_{[k]}$, $BDFM_{[k+1]}$ or our space $\mathcal{S}_k$, then we have*

$$\|\mathbf{u}-\mathbf{u}_h\|_{g_{k+1},\Omega} \le Ch^{k+2}|\mathbf{u}|_{k+2} \tag{3.8}$$

$$||\mathbf{u}_h-\Pi_h\mathbf{u}||_{0,\Omega} \le Ch^{k+2}|\mathbf{u}|_{k+2}. \tag{3.9}$$

We now present some results for scalar variable. First let Φ_h be the L^2-projection onto the space W_h.

THEOREM 3.2. *If $p \in H^{k+2}(\Omega)$ and $\mathbf{u} \in \mathbf{H}^{k+2}(\Omega)$ are the solutions of (2.2), and if $(\mathbf{u}_h, p_h)$ are the solutions of (2.4) for $RT_{[k]}$ space, then we have*

$$\|p_h-\Phi_h p\|_0 \le Ch^{k+2}|\mathbf{u}|_{k+2} \tag{3.10}$$

$$\|p-p_h\|_{g_{k+1}} \le Ch^{k+2}|p|_{k+2}. \tag{3.11}$$

Proof. The first result is given in [11, 14] and the second one is in [16]. ◻

REMARK 3.1. *Since our pressure space $\mathcal{R}_{k-1}$ contains the pressure space W_h of $BDFM_{[k+1]}$, the proof in [11] carries over to our space and the result (3.10) also holds for our space with a minor modification. But the result of type (3.11) holds only for $RT_{[k]}$ space.*

3.1. Post-processing. This post processing technique is similar to [21] and is primarily intended to improve suboptimal convergence of scalar variable for quadrilateral case. But for the case of rectangular element, we obtain superconvergence. (One order higher than the result in [21] for $k = 1$.)

Given the solution $(\mathbf{u}_h, p_h)$ of (2.4), we define a new pressure solution, $p_h^{\#} \in \tilde{W}_h$ locally on each element $K \in \mathscr{T}_h$ as follows:

$$\int_K \kappa\nabla p_h^{\#}\cdot\nabla q\,d\mathbf{x} = -\int_K \mathbf{u}_h\cdot\nabla q\,d\mathbf{x}, \quad \forall q \in \tilde{W}_h(K), \tag{3.12}$$

$$\int_K p_h^{\#}\,d\mathbf{x} = \int_K p_h\,d\mathbf{x} \tag{3.13}$$

where

$$\tilde{W}_h(K) = \begin{cases} Q_{1,1}(K) & \text{for } k=1,\\ P_{k+1}(K) & \text{for } k\ge 2.\end{cases}$$

REMARK 3.2. *We could have used $P_{k+1}(K)$ in the definition of $\tilde{W}_h(K)$ for all $k \ge 1$. But since the smaller space $Q_{1,1}(K)(\subsetneq P_2(K))$ works for the optimal order* *when $k = 1$, we replaced it by the smaller space $Q_{1,1}(K)$. Instead, the proof requires a special treatment, which we present separately below.*

Now we prove superconvergence of $p_h^{\#}$ at Gauss points. Let $\tilde{\Phi}_K$ denote the L^2-projection onto $\tilde{W}_h(K)$ for all $k \geq 1$. In particular, when $k = 1$, $\tilde{\Phi}_K$ is the same as Φ_K, the L^2-projection onto $Q_{1,1}(K)$. In general, $\tilde{\Phi}_K$ is the L^2-projection onto $P_{k+1}(K)$. Then we have

Lemma 3.3. *If $p \in H^3(K)$ and $\psi \in Q_{1,1}(K)$, then we have*

$$| \langle\langle \nabla(\Phi_K p - p), \nabla\psi \rangle\rangle_{g_2} | \leq Ch^2 \|\nabla\psi\|_{g_2} |p|_{H^3(K)}. \tag{3.14}$$

Proof. Let $\hat{K} = [-1,1] \times [-1,1]$ be the reference element. For a fixed $\psi \in Q_{1,1}(\hat{K})$ define a linear functional on, say $H^{k+1}(\hat{K})$ by

$$f_\psi(\hat{p}) = \left\langle\left\langle \nabla(\Phi_{\hat{K}}\hat{p} - \hat{p}), \nabla\psi \right\rangle\right\rangle_{g_{k+1}}. \tag{3.15}$$

Then it is clear that the norm of this functional satisfies

$$\|f_\psi\|^* \leq C|\psi|_{1,\hat{K}}. \tag{3.16}$$

We show that this functional vanishes for $\hat{p} \in P_2(\hat{K})$. Since $\Phi_{\hat{K}}$ preserves functions in $Q_{1,1}(\hat{K})$, it suffices to assume $\hat{p} \in \{\hat{x}^2, \hat{y}^2\}$. First we assume $\hat{p} = \hat{x}^2$. The case when $\hat{p} = \hat{y}^2$ can be handled exactly the same way. Since the Legendre polynomials on $[-1,1]$ are

$$\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\hat{x}, \sqrt{\frac{5}{4}}(3\hat{x}^2 - 1), \cdots,$$

we see $\Phi_{\hat{K}}(\hat{x}^2) = c_0$ for some constant and hence $\nabla(\Phi_{\hat{K}}(\hat{x}^2)) = 0$. Thus

$$\left\langle\left\langle \nabla(\Phi_{\hat{K}}\hat{p} - \hat{p}), \nabla\psi \right\rangle\right\rangle_{g_2} = \langle\langle -\nabla\hat{p}, \nabla\psi \rangle\rangle_{g_2}.$$

Now we claim

$$\langle\langle -\nabla\hat{p}, \nabla\psi \rangle\rangle_{g_2} = 0. \tag{3.17}$$

Since $\nabla\hat{p} = (2\hat{x}, 0)^T$ and the first component of $\nabla\psi$ is a linear function of $\hat{y}$ when $\psi \in Q_{1,1}$. Thus (3.17) vanishes as an integral of odd function on symmetric interval. Now Bramble-Hilbert lemma shows

$$|f_\psi(\hat{p})| \leq \|f_\psi\|^* |\hat{p}|_{3,\hat{K}}.$$

Now transformation back to K and use (3.16) to obtain the appropriate order :

$$|f_\psi(p)| \leq Ch^2 \|f_\psi\|^* |p|_{3,K} \leq Ch^2 \|\nabla\psi\|_{g_2} |p|_{3,K}.$$

□

Lemma 3.4. *We have that for $k \geq 2$,*

$$\|\nabla(\tilde{\Phi}_h p) - \nabla p\|_{0,K} \leq Ch^{k+1} |p|_{k+2,K}. \tag{3.18}$$

The same result also holds when the L^2-norm is replaced by Gauss norm $\|\cdot\|_{g_{k+1}}$.

Proof. First we consider the following functional on a reference element $\hat{K}$: Define

$$f(\hat{p}) = \|\nabla(\tilde{\Phi}_K\hat{p}) - \nabla\hat{p}\,\|_{0,\hat{K}}.$$

Then this functional vanishes on P_{k+1}. Hence by Bramble-Hilbert lemma, we have that

$$\|\nabla(\tilde{\Phi}_K\hat{p}) - \nabla\hat{p}\,\|_{0,\hat{K}} \le C|\hat{p}\,|_{k+2,\hat{K}}.$$

Now we apply the scaling argument.

$$\|\nabla(\tilde{\Phi}_K p) - \nabla p\,\|_{0,K} \le Ch^{k+1}|p\,|_{k+2,K}. \tag{3.19}$$

Therefore, the proof is complete. □

LEMMA 3.5. *Let $q = \tilde{\Phi}_K p - p_h^{\#}$. Then we have for $k \ge 1$.*

$$|q|_{1,K} = \|\nabla q\|_{g_{k+1}} \le Ch^{k+1}(|p\,|_{k+2,K} + h|\mathbf{u}|_{k+2,K}). \tag{3.20}$$

Proof. We have

$$\begin{aligned}
|q|_{1,K}^2 = \|\nabla q\|_{g_{k+1}}^2 &= \left\langle\!\left\langle \nabla(\tilde{\Phi}_K p - p_h^{\#}), \nabla q \right\rangle\!\right\rangle_{g_{k+1}} \\
&= \left\langle\!\left\langle \nabla(\tilde{\Phi}_K p - p), \nabla q \right\rangle\!\right\rangle_{g_{k+1}} + \left\langle\!\left\langle \nabla(p - p_h^{\#}), \nabla q \right\rangle\!\right\rangle_{g_{k+1}} \\
&= \left\langle\!\left\langle \nabla(\tilde{\Phi}_K p - p), \nabla q \right\rangle\!\right\rangle_{g_{k+1} + \langle\langle \kappa^{-1}\kappa\nabla(p-p_h^{\#}),\nabla q\rangle\rangle_{g_{k+1}}} \\
&= \left\langle\!\left\langle \nabla(\tilde{\Phi}_K p - p), \nabla q \right\rangle\!\right\rangle_{g_{k+1}} + \left\langle\!\left\langle (-\mathbf{u} + \mathbf{u}_h)\,, \kappa^{-1}\nabla q \right\rangle\!\right\rangle_{g_{k+1}}
\end{aligned} \tag{3.21}$$

When $k = 1$ we have, by (3.8), (3.14),

$$\|\nabla q\|_{g_2}^2 \le Ch^2\|p\,\|_{3,K}\|\nabla q\|_{g_2} + Ch^3|\mathbf{u}|_3\|\nabla q\|_{g_2},$$

while for $k > 1$ we obtain by (3.8) and (3.18).

$$\|\nabla q\|_{g_{k+1}}^2 \le Ch^{k+1}(\|p\,\|_{k+2,K}\|\nabla q\|_{g_{k+1}} + h|\mathbf{u}|_{k+2,K}\|\nabla q\|_{g_{k+1}}).$$

This completes the proof. □

THEOREM 3.6. *We have for $k \ge 1$*

$$\|p - p_h^{\#}\|_{g_{k+1}} \le Ch^{k+2}(|p\,|_{k+2} + \|\mathbf{u}\|_{k+2}). \tag{3.22}$$

Proof. By triangle inequality we have

$$\|p - p_h^{\#}\|_{g_{k+1}} \le \|p - \tilde{\Phi}_K p\,\|_{g_{k+1}} + \|\tilde{\Phi}_K p - p_h^{\#}\|_{g_{k+1}}. \tag{3.23}$$

We estimate $q = \tilde{\Phi}_K p - p_h^{\#}$ with $\bar{q} = \frac{1}{\text{Area}(K)}\int_K q\,d\mathbf{x}$. Since $(p_h^{\#}, 1)_K = (p_h, 1)_K$, we have

$$\begin{aligned}
\|\bar{q}\|_\infty &= \left|\frac{1}{Area(K)}\int_K (\tilde{\Phi}_K p - p_h^{\#})d\mathbf{x}\right| = \left|\frac{1}{Area(K)}\int_K (\tilde{\Phi}_K p - p_h)d\mathbf{x}\right| \\
&\le Ch^{-1}\|\tilde{\Phi}_K p - p_h\|_{0,K}.
\end{aligned}$$

Hence

$$\|\bar{q}\|_{g_{k+1}} \leq Ch\|\bar{q}\|_\infty \leq C\|\tilde{\Phi}_K p - p_h\|_{0,K}. \tag{3.24}$$

By Poincaré inequality, we have by Lemma 3.5

$$\begin{aligned}\|q\|_{g_{k+1}} &\leq \|q-\bar{q}\|_{g_{k+1}} + \|\bar{q}\|_{g_{k+1}} \\ &\leq Ch|q|_{1,K} + \|\bar{q}\|_{g_{k+1}} \\ &\leq Ch^{k+2}(|p\,|_{k+2} + |\mathbf{u}|_{k+2}) + C\|\tilde{\Phi}_K p - p_h\|_{0,K}.\end{aligned} \tag{3.25}$$

Now by (3.10), (3.23) through (3.25), the proof is complete.
□

4. Numerical Experiment. In this section we present some numerical experiments when $k=1$. Throughout the numerical computations we solve problem (2.4) on the unit square $\bar{\Omega} = [0,1]\times[0,1]$.

As a first example, we let $\kappa = I_M$, the identity matrix, and for the second we let

$$\kappa = \begin{pmatrix} 1+10x+y & 0 \\ 0 & 1+10x+y \end{pmatrix}. \tag{4.1}$$

In these two examples, $p(x,y) = (x-x^2)(y-y^2)$ is the exact solution. As a third example, we choose a (scalar) discontinuous coefficient:

$$\kappa = \begin{cases} 1000, & \text{for } x,y \geq 0.5 \\ 1, & \text{otherwise,} \end{cases} \tag{4.2}$$

with the exact solution $p(x,y) = \left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)\sin(2\pi x)\cos\left(2\pi y + \frac{1}{2}\pi\right)/\kappa$.

Table 4.1
The result with $\kappa = I_M$.

	$\|p-p_h\|_{g_2}$		$\|\mathbf{u}-\mathbf{u}_h\|_{g_2}$		$\|p-p_h^{\#}\|_{g_2}$	
n	error	order	error	order	error	order
4	1.03097e-03		4.74286e-09		1.05865e-13	
8	2.70631e-04	1.939	4.80646e-09	*	4.74964e-13	*
16	6.84632e-05	1.982	4.34370e-09	*	4.25500e-13	*
32	1.71661e-05	1.995	4.37379e-09	*	3.51110e-12	*
64	4.29468e-06	1.998	7.39267e-09	*	8.18826e-11	*

In each table, the first two columns show the results of pressure and velocity of our scheme without post processing, while the third column is the result of scalar variable after post-processing.

The numerical results for the case $\kappa = I_M$ are shown in Table 4.1. For this particular example, we got the solution exact up to machine accuracy. Table 4.2 show an excellent agreement with the theory while table 4.3 shows the result for a problem with discontinuous coefficients, in which case the result also matches with the theory.

TABLE 4.2
The results of convergence with κ in (4.1).

n	$\|p-p_h\|_{g_2}$ error	order	$\|\mathbf{u}-\mathbf{u}_h\|_{g_2}$ error	order	$\|p-p_h^{\#}\|_{g_2}$ error	order
4	1.03088e-03		1.66751e-03		1.87073e-04	
8	2.70624e-04	1.929	2.39969e-04	2.796	2.54441e-05	2.878
16	6.84629e-05	1.982	3.16883e-05	2.920	3.27311e-06	2.958
32	1.71661e-05	1.995	4.03274e-06	2.974	4.12549e-07	2.988
64	4.29468e-06	1.998	5.08067e-07	2.988	5.16852e-08	2.996

TABLE 4.3
The results of convergence order with κ in (4.2).

n	$\|p-p_h\|_{g_2}$ error	order	$\|\mathbf{u}-\mathbf{u}_h\|_{g_2}$ error	order	$\|p-p_h^{\#}\|_{g_2}$ error	order
4	3.48900e-03		2.59183e-02		1.57773e-03	
8	1.25417e-03	1.476	2.95180e-03	3.134	2.27154e-04	2.796
16	3.32170e-04	1.916	3.14327e-04	3.231	2.63734e-05	3.106
32	8.42043e-05	1.979	3.69582e-05	3.088	3.16894e-06	3.057
64	2.11242e-05	1.994	4.54093e-06	3.024	3.91351e-07	3.017

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