NEW POISSON-BOLTZMANN TYPE EQUATIONS: ONE-DIMENSIONAL SOLUTIONS

CHIUN-CHANG LEE *, HIJIN LEE †, YUNKYONG HYON ‡, TAI-CHIA LIN §, AND CHUN LIU¶

Abstract. The Poisson-Boltzmann (PB) equation is conventionally used to model the equilibrium of *bulk* ionic species in different media and solvents. In this paper we study a new Poisson-Boltzmann type (PB_n) equation with a small dielectric parameter ϵ^2 and nonlocal nonlinearity which takes into consideration of the preservation of the total amount of each individual ion. This equation can be derived from the original Poisson-Nernst-Planck (PNP) system. Under Robin type boundary conditions with various coefficient scales, we demonstrate the asymptotic behaviors of one dimensional solutions of PB_n equations as the parameter ϵ approaches to zero. In particular, we show that in case of electroneutrality, i.e., $\alpha = \beta$, solutions of 1-D PB_n equations have the similar asymptotic behavior as those of 1-D PB equations. However, as $\alpha \neq \beta$ (non-electroneutrality), solutions of 1-D PB_n equations may have blow-up behavior which can not be found in 1-D PB equations. Such a difference between 1-D PB and PB_n equations can also be verified by numerical simulations.

Key words. Poisson-Nernst-Planck system, new Poisson-Boltzmann type equations

AMS subject classifications. 76A05, 76M99, 65C30

1. Introduction.

Understanding ion transport is crucial in the study of many physical and biological problems, such as semiconductors [35], electro-kinetic fluids [32, 37, 38, 39], transport of electrochemical systems [36] and ion channels in cell membranes [8, 16, 17, 26, 28, 31, 33, 34]. One of the fundamental models for the ionic transport is the time dependent system consisting coupled diffusion-convection equations, the Poisson-Nernst-Planck (PNP) system.

In this paper, we confine ourselves to the physical domain as a one-dimensional interval $(-1, 1)$, as was done in [1]. The PNP system is given by

$$n_t = -\partial_x J_n, \qquad p_t = -\partial_x J_p, \tag{1.1}$$

$$J_n = -D_n \left(n_x - \frac{z_n e}{k_B T} n \phi_x \right), \quad J_p = -D_p \left(p_x + \frac{z_n e}{k_B T} p \phi_x \right), \tag{1.2}$$

$$\epsilon^2 \phi_{xx} = -\rho + z_n e n - z_p e p, \qquad \text{for } x \in (-1, 1),\ t > 0 \tag{1.3}$$

where ϕ is the electrostatic potential, n is the density of anions, p is the density of cations, ρ is the permanent (fixed) charge density in the domain, z_n, z_p are the valence

*Department of Mathematics, National Taiwan University, No.1, Sec.4, Roosevelt Road, Taipei 106, Taiwan, email:f92221030@ntu.edu.tw ,

†Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology, Daejeon, Republic of Korea 305-701, email:hijin@kaist.ac.kr,

‡Institute for Mathematics and Its Applications, 114 Lind, 207 Church St. S.E., Minneapolis, MN 55455, USA, email: hyon@ima.umn.edu,

§Department of Mathematics, National Taiwan University, Taida Institute for Mathematical Sciences (TIMS), and National Center for Theoretical Sciences (Taipei Office), No.1, Sec.4, Roosevelt Road, Taipei 106, Taiwan, email: tclin@math.ntu.edu.tw,

¶Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA, and Institute for Mathematics and Its Applications, University of Minnesota, 114 Lind, 207 Church St. S.E., Minneapolis, MN 55455, USA, email: liu@math.psu.edu, liu@ima.umn.edu

of ions, e is the elementary charge, k_B is the Boltzmann constant, T is temperature, J_n, J_p are the ionic flux densities and D_n, D_p are their diffusion coefficients. The parameter $\epsilon = \left(\epsilon_0\, U_T/(d^2\, e\, S)\right)^{1/2} > 0$, where ϵ_0 is the dielectric constant of the electrolyte, U_T is the thermal voltage, d is the length of the domain $(-1, 1)$, and S is the appropriate concentration scale (cf. [20]). Furthermore, $\epsilon\, d$ is known as the Debye length and ϵ is of order 10^{-2} for the physiological cases of interest (cf. [2]). Thus we may assume ϵ as a small parameter tending to zero. For simplicity, we consider monovalent ions, that is, $z_n = z_p = 1$ with $e/k_BT = 1$, $\rho = 0$, $D_n = D_p = 1$. Here we reuse the notation, ϵ again. Then the PNP system (1.1)-(1.3) becomes

$$n_t = -\partial_x J_n\,, \qquad p_t = -\partial_x J_p, \tag{1.4}$$

$$J_n = -(n_x - n\phi_x)\,, \quad J_p = -(p_x + p\phi_x), \tag{1.5}$$

$$\epsilon^2 \phi_{xx} = n - p\,, \qquad \text{for } x \in (-1, 1),\ t > 0. \tag{1.6}$$

The equation (1.4) represents the conservation of total charges of the individual ions (similar to the conservation of mass equation in [18]). These so-called Nernst-Planck equations describe electro-diffusion and electrophoresis according to Fick's and Kohlrausch's laws, respectively. The equation (1.6) depicts the electro-static Poisson's law. We assume no-flux boundary conditions on the boundary to describe the insulated domain boundaries:

$$J_n = J_p = 0 \quad \text{for } x = \pm 1,\ t > 0. \tag{1.7}$$

Hence it is easy to conclude that the total charges of both negative and positive ions are conserved in time by (1.4) and (1.7):

$$\frac{d}{dt}\int_{-1}^{1} n dx = -\int_{-1}^{1} \partial_x J_n dx = -J_n\big|_{x=-1}^{x=1} = 0,$$
$$\frac{d}{dt}\int_{-1}^{1} p dx = -\int_{-1}^{1} \partial_x J_p dx = -J_p\big|_{x=-1}^{x=1} = 0, \quad \text{for } t > 0.$$

Consequently, we have the normalization condition given by

$$\int_{-1}^{1} n\, dx = \alpha, \quad \int_{-1}^{1} p\, dx = \beta, \quad \text{for } t > 0 \tag{1.8}$$

where α and β are positive constants only determined by the initial conditions.

The PNP system (1.4)-(1.6) can be derived by the energetic variational approaches [37, 40]. The energy dissipation law of the PNP system (1.4)-(1.6) is given as follows:

$$\frac{d}{dt}\int_{-1}^{1}\left(n\log n + p\log p + \frac{\epsilon^2}{2}|\phi_x|^2\right)dx = -\int_{-1}^{1}\left(n\left|\frac{n_x}{n} - \phi_x\right|^2 + p\left|\frac{p_x}{p} + \phi_x\right|^2\right)dx.$$

Here the integral of $n\log n$ and $p\log p$ is the entropy related to Brownian motion of anions and cations, and the integral of $\frac{\epsilon^2}{2}|\phi_x|^2$ is the electrostatic potential of the coulomb interaction between charged ions.

To see the equilibrium of the PNP system (1.4)-(1.6), it is standard to consider the steady state i.e. time-independent solutions of the PNP system (1.4)-(1.6). Such a concept is well-known in many fields including biological systems and the electrophysiology. Hence we may set $n_t \equiv p_t \equiv 0$ and (1.4)-(1.6) can be transformed into the following system

$$\partial_x(n_x - n\phi_x) = 0, \quad \partial_x(p_x + p\phi_x) = 0, \quad \text{for} \quad x \in (-1,1), \tag{1.9}$$
$$\epsilon^2 \phi_{xx} + p - n = 0, \quad \text{for} \quad x \in (-1,1) \tag{1.10}$$

where $n = n(x)$, $p = p(x)$, $\phi = \phi(x)$ are real-valued functions. On the other hand, the no-flux boundary condition (1.7) and the normalization condition (1.8) become

$$(n_x - n\phi_x)(\pm 1) = 0, \quad (p_x + p\phi_x)(\pm 1) = 0\,, \tag{1.11}$$

and $\int_{-1}^{1} n dx = \alpha$, $\quad \int_{-1}^{1} p dx = \beta$ where α and β are positive constants.

Remark 1.1. *We want to point out, although most of the physical and biological systems possess the (overall) electroneutrality, it may not be the case in local domains (cf. [9, 15, 21]), which are the considerations of this paper. Such non-electroneutrality phenomena are ubiquitous and can be extremely subtle when systems involves empirical boundary conditions to take into account of electrochemical reactions on the boundaries, such as those in [3, 4, 5, 19]. For instance, the non-electroneutrality may hold near the electrodes when the Faradaic current is driven by redox reactions occurring at the electrodes (cf. [46]).*

Boundary conditions play a crucial role on solutions of the system (1.9), (1.10). The boundary conditions (1.11) guarantee the solutions of (1.9) have the following forms:

$$n = n(x) = \tilde{\alpha} e^{\phi(x)}, \quad p = p(x) = \tilde{\beta} e^{-\phi(x)}, \quad \text{for } x \in (-1,1) \tag{1.12}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are constants. Note that from (1.9), we have $n_x - n\phi_x \equiv c_1$ and $p_x + p\phi_x \equiv c_2$ for some constants c_j's. Then the conditions (1.11) imply $c_1 = c_2 = 0$ which gives (1.12) by solving $n_x - n\phi_x \equiv 0$ and $p_x + p\phi_x \equiv 0$, respectively. To fulfill the total charge condition, $\tilde{\alpha}$ and $\tilde{\beta}$ must satisfy

$$\tilde{\alpha} = \frac{\alpha}{\int_{-1}^{1} e^{\phi} dx}\,, \quad \tilde{\beta} = \frac{\beta}{\int_{-1}^{1} e^{-\phi} dx} \tag{1.13}$$

having the nonlocal dependence on ϕ. Consequently, (1.10), (1.12) and (1.13) give the new Poisson-Boltzmann type (PB_n) equation,

$$\epsilon^2 \phi_{xx} = \frac{\alpha\, e^{\phi}}{\int_{-1}^{1} e^{\phi} dx} - \frac{\beta\, e^{-\phi}}{\int_{-1}^{1} e^{-\phi} dx} \quad \text{for } x \in (-1,1). \tag{1.14}$$

Conventionally, the Poisson-Boltzmann (PB) equation for electrolytes with two species' densities n and p occupying $(-1,1)$ is the combination of the Poisson equation

$$\epsilon^2 \phi_{xx} = n - p \quad \text{in } (-1,1)\,,$$

and the Boltzmann distribution

$$n = \alpha_1\, e^{\phi} \quad \text{and} \quad p = \beta_1\, e^{-\phi} \quad \text{in } (-1,1)\,,$$

where α_1 and β_1 are positive constants (cf. [7], [11] and [12]). Then the PB equation can be written as

$$\epsilon^2 \phi_{xx} = \alpha_1 e^{\phi} - \beta_1 e^{-\phi}, \quad \text{for } x \in (-1, 1) . \tag{1.15}$$

Note that the PB equation (1.15) has only (spatially) local nonlinearity which is very different from the nonlocal nonlinearity of the PB_n equation (1.14). We may derive the PB equation (1.15) from the steady state PNP system (1.9)-(1.10) with another boundary conditions as follows:

$$(n_x - n\phi_x)(-1) = (p_x + p\phi_x)(-1) = 0, \tag{1.16}$$

$$n(1) = n_1, \quad p(1) = p_1, \quad \phi(1) = \phi_1 \tag{1.17}$$

where n_1, p_1 and ϕ_1 are positive constants satisfying $\alpha_1 = n_1\, e^{-\phi_1}$ and $\beta_1 = p_1\, e^{\phi_1}$. One may remark that both the PB_n equation (1.14) and the PB equation (1.15) come from the same PNP system (1.9)-(1.10) but with respect to the different boundary conditions (1.11) and (1.16)-(1.17), respectively. Physically, the boundary conditions (1.11) provide no-flux at the boundary points $x = \pm 1$ but the boundary conditions (1.16)-(1.17) have the Dirichlet boundary condition which may give nonzero flux at the one end $x = 1$ and no-flux at the other end $x = -1$. Since various boundary conditions result in distinct solution behaviors, it would be natural to believe that the PB_n equation (1.14) and the PB equation (1.15) have different solution behaviors.

In many applications, especially those in biological systems, the interfacial boundary condition is extremely important. Here we may consider the interval $(-1, 1)$ as the ion channel with boundaries at $x = \pm 1$. To describe more general physical phenomena, a Robin-type boundary condition (cf. [10]) for the electrostatic potential ϕ at $x = \pm 1$ is given by

$$\eta_\epsilon \frac{\partial \phi}{\partial \nu} = \phi_{extra} - \phi_{intra} \tag{1.18}$$

where ν is the unit outer normal vector, and ϕ_{intra} and ϕ_{extra} are the intrachannel and extrachannel electrostatic potentials at the channel boundaries, respectively. The coefficient $\eta_\epsilon \sim \frac{\epsilon_0}{\epsilon_m}$ is governed by the ratio of ϵ_0 the dielectric constant of the electrolyte solution and ϵ_m the dielectric constant of the membrane (cf. [44, 45]). Due to $\epsilon_0 \sim \epsilon^2$, η_ϵ depends on the parameter ϵ and satisfies $\eta_\epsilon/\epsilon \sim \epsilon/\epsilon_m$. Generically, one may not know the asymptotic limit of the ratio η_ϵ/ϵ as the parameter ϵ goes to zero. Here we assume the limit $\lim_{\epsilon\to 0} \eta_\epsilon/\epsilon$ to be a nonnegative constant γ and infinity, respectively.

The boundary condition (1.18) may also represent the capacitance effect of cell membrane [23, 24, 25] and can be found in other physical systems e.g. [37, 38, 39]. Setting $\phi_{extra} = \phi_0$ and $\phi_{intra} = \phi$, we may transform the condition (1.18) into

$$\begin{cases} (\phi + \eta_\epsilon \phi_x)(x) & = \phi_0(x), \quad \text{if } x = 1, \\ (\phi - \eta_\epsilon \phi_x)(x) & = \phi_0(x), \quad \text{if } x = -1 \end{cases} \tag{1.19}$$

where $\phi_0(1), \phi_0(-1)$ are constants and η_ϵ is a nonnegative constant, with scale of length, and related to the surface capacitance. Since we want to study the properties of the solutions with respect to the relative relation of this parameter and the dielectric constant, we assume that η_ϵ depends on the parameter ϵ. Notice, as $\eta_\epsilon = 0$, (1.19) becomes the Dirichlet boundary condition. As $\eta_\epsilon \to \infty$, (1.19) tends to the Neumann boundary condition. It is obvious that the Robin boundary condition (1.19) has more general feature rather than Dirichlet and Neumann boundary conditions.

The main goal of this paper is to compare solutions of the PB_n equation (1.14) and the PB equation (1.15) under the general Robin boundary condition (1.19) for the electrostatic potential. For the coefficients α, β of the PB_n equation (1.14), two cases are considered as follows: One case is $\alpha = \beta$ for the global electroneutrality, which means the total amounts of the positive charge and the negative charge are the same. The other case is $\alpha \neq \beta$ for the non-electroneutrality which means the total positive and negative charge densities are not equal to each other. For the later case, we want emphasize again, although most of the physical and biological systems possess the (overall) electroneutrality, it may not be the case in local domains (cf. [9, 15, 21]), which are the considerations of this paper. For these two cases, we prove rigorously interior and boundary estimates for the PB_n equation (1.14) with the boundary condition (1.19):

- Under the boundary condition (1.19), if $\alpha = \beta$, then the solution of (1.14) asymptotically close to the solution of (1.15) with $\alpha_1 = \alpha/2 = \beta/2 = \beta_1$ as ϵ goes to zero. This shows that as $\alpha = \beta$, solutions of the PB_n equation (1.14) and the PB equation (1.15) may have the same asymptotic behavior.

- If $\alpha \neq \beta$, then the solution of (1.14) with the boundary condition (1.19) may tend to infinity as ϵ goes to zero. Such a blow-up behavior is absent for the solutions of (1.15) with the boundary condition (1.19). This provides one of the key different behaviors of the solutions for PB_n equation (1.14) and the PB equation (1.15).

Now we summarize our results as follows:

(a) In the electroneutral case ($\alpha = \beta$):

(a1) If $\lim_{\epsilon \downarrow 0} \frac{\epsilon}{\eta_\epsilon} = 0$, the solution ϕ approaches zero in $[-1, 1]$ as $\epsilon \downarrow 0$. However, ϕ has slope of order $O(1/\eta_\epsilon)$ on the boundary (cf. Theorem 1.3(i)).

(a2) When $\frac{\epsilon}{\eta_\epsilon} \geq C$ for some positive constant C independent of ϵ, the solution ϕ possesses boundary layers with thickness ϵ (cf. Theorem 1.3 (ii) and (1.24)).

(b) In the non-electroneutral case ($\alpha \neq \beta$):
The solution ϕ has boundary layers with thickness ϵ^2 and $\phi(x) - \phi(\pm 1)$ tends to infinity with the leading order term $\log(\epsilon^{-2})$ as $\epsilon \downarrow 0$ for $x \in (-1, 1)$ (see Theorem 1.5 – 1.7). The values $\phi(\pm 1)$ can be estimated as follows:

(b1) If $\frac{\eta_\epsilon}{\epsilon^2} \leq C$, $\phi(1)$ and $\phi(-1)$ converge to different finite values as $\epsilon \downarrow 0$, where C is a positive constant independent of ϵ (cf. Theorem 1.6(i) and (ii)).

(b2) If $\lim_{\epsilon \downarrow 0} \frac{\eta_\epsilon}{\epsilon^2} = \infty$, both $\phi(1)$ and $\phi(-1)$ diverge to ∞, but $|\phi(1) - \phi(-1)|$ converges to zero as $\epsilon \downarrow 0$ (cf. Theorem 1.6(iii)).

(c) The difference between the solutions to the PB_n equation (1.14) and the PB equation (1.25) can be stated as follows:

(c1) When $\alpha = \beta$, the solution of the PB_n equation (1.14) may converge to the solution of the PB equation (1.25) (cf. Theorem 1.4). Namely, in the case of $\alpha = \beta$, the solution of the PB_n equation (1.14) have the same asymptotic behavior as that of the PB equation (1.25).

(c2) When $\alpha \neq \beta$, the solution of the PB equation (1.25) remain bounded for $\epsilon > 0$ (cf. Theorem 4.2). However, as $\alpha \neq \beta$, the solution of the PB_n equation (1.14) may tend to infinity as ϵ goes to zero (see **(b)**). This may provide the difference between the solutions to the PB_n equation (1.14) and the PB equation (1.25).

Although people may believe that the non-electroneutrality i.e. $\alpha \neq \beta$ is impossible for most biological systems, it seems difficult to specify what kind of biological systems making the non-electroneutrality impossible. Here we show that if the non-electroneutrality holds, then the blow-up behavior of the electrical potential ϕ occurs by solving the new model PB_n equation (1.14). Thus the non-electroneutrality is impossible for those biological systems without the blow-up behavior of the electrical potential. To avoid the non-electroneutrality, the new model PB_n equation (1.14) provides a new way which can not be found in the conventional PB equation (1.15).

Finally, we want to point out that in our ongoing projects [13, 14], we studied the PB_n equations with multiple species, such as those of Na^+, K^+, Ca^{2+}, Cl^-, as in the real biological environment. We also study the corresponding results for higher dimensional cases.

1.1. Main Theorems. Hereafter, we set $\phi = \phi(x)$ as the solution of the equation (1.14) with the boundary condition (1.19) [22, 27, 29, 30]. The solution ϕ may depend on the parameter ϵ and should be denoted as ϕ_ϵ but we denote it as ϕ for simplicity. The equation (1.14) with the boundary condition (1.19) can be regarded as the Euler-Lagrange equation of the energy functional written as

$$\begin{aligned} E_\epsilon[u] =& \frac{\epsilon^2}{2} \int_{-1}^{1} |u'|^2 \, dx + \alpha \log \left(\int_{-1}^{1} e^u \, dx \right) + \beta \log \left(\int_{-1}^{1} e^{-u} \, dx \right) \\ &+ \frac{\epsilon^2}{2\eta_\epsilon^2} \left[(\phi_0(1) - u(1))^2 + (\phi_0(-1) - u(-1))^2 \right], \end{aligned} \tag{1.20}$$

for $u \in H^1((-1,1))$. Using the standard Direct method, we get the energy minimizer of E_ϵ over $H^1((-1,1))$. Applying the standard elliptic regularity theory (cf. [6]), we obtain the classical solution of (1.14) with the boundary condition (1.19). The uniqueness theorem is stated as follows:

THEOREM 1.1. *(Uniqueness Theorem) There exists at most one* $\phi \in C^\infty((-1,1)) \cap C^2([-1,1])$ *solution of the equation (1.14) with the boundary condition (1.19).*
We will give the proof of Theorem 1.1 in Appendix (Section 6).

Suppose $\alpha = \beta$. When $\phi_0(-1) = \phi_0(1) = A$, one gets $\phi \equiv A$ trivially by Theorem 1.1. Note that the equation (1.14) is invariant under the replacement of ϕ by $\phi + C$, for any nonzero constant C. Hence for the case of $\phi_0(-1) \neq \phi_0(1)$, without loss of generality, we may assume $\phi_0(-1) = -\phi_0(1) > 0$ by adjusting a suitable constant C. Such an assumption and Theorem 1.1 can assure the solution ϕ is an odd function on $[-1,1]$ i.e., $\phi(-x) = -\phi(x), \forall x \in [-1,1]$. Consequently, we have $\int_{-1}^{1} e^{\phi} dx = \int_{-1}^{1} e^{-\phi} dx$ which is crucial to study the equation (1.14). Now we state the interior estimate of ϕ as follows:

THEOREM 1.2. *(Interior Estimates) Assume* $\alpha = \beta > 0$ *and* $\phi_0(1) = -\phi_0(-1) > 0$. *For* $\epsilon > 0$, *let* $\phi \in C^\infty((-1,1)) \cap C^2([-1,1])$ *be the solution of the equation (1.14) with the boundary condition (1.19). Then the solution* ϕ *is odd, monotonically increasing on* $[-1,1]$, *convex on* $(0,1)$, *and concave on* $(-1,0)$. *Moreover, there exists* $M > 0$ *independent of* ϵ *such that*

$$|\phi(x)| \leq \phi_0(1) \left(e^{-\frac{M}{\epsilon}(1+x)} + e^{-\frac{M}{\epsilon}(1-x)} \right), \quad \forall\, x \in [-1,1],\ \epsilon > 0. \tag{1.21}$$

At the boundary $x = 1$, the asymptotic behavior of $\phi(x)$ may depend on η_ϵ. Here we study two cases for $\eta_\epsilon > 0$: (i) $\frac{\eta_\epsilon}{\epsilon} \to \infty$ as $\epsilon \downarrow 0$, (ii) $\frac{\eta_\epsilon}{\epsilon} \to \gamma < \infty$ as $\epsilon \downarrow 0$. The boundary estimates of these cases are presented as follows:

THEOREM 1.3. *(Boundary Estimates) Under the same hypotheses as in Theorem 1.2, we have*

(i) if $\frac{\eta_\epsilon}{\epsilon} \to \infty$ *as* $\epsilon \downarrow 0$, *then* $\lim_{\epsilon\downarrow 0} |\phi(x)| = 0$ *uniformly in* $[-1, 1]$, *and* $\lim_{\epsilon\downarrow 0} \eta_\epsilon \phi'(1) = \phi_0(1)$.

(ii) if $\frac{\eta_\epsilon}{\epsilon} \to \gamma$ *as* $\epsilon \downarrow 0$, *where* $0 \le \gamma < \infty$, *then*

$$\lim_{\epsilon\downarrow 0} \phi(1) = \phi^* \quad \text{and} \quad \lim_{\epsilon\downarrow 0} \epsilon\phi'(1) = \sqrt{\alpha}\left(e^{\phi^*/2} - e^{-\phi^*/2}\right) \tag{1.22}$$

where $0 < \phi^* \le \phi_0(1)$ *is uniquely determined by*

$$\phi_0(1) - \phi^* = \gamma\sqrt{\alpha}\left(e^{\phi^*/2} - e^{-\phi^*/2}\right). \tag{1.23}$$

For the case (ii), the asymptotic behavior of ϕ *near the boundary* $x = 1$ *can be represented by*

$$\phi_1^\epsilon(x) \le \phi(x) \le \phi_2^\epsilon(x) \quad \text{and} \quad \phi_i^\epsilon(x) = 2\log\left(\frac{a_i^\epsilon + b_i^\epsilon e^{-\frac{c_i^\epsilon}{\epsilon}(1-x)}}{a_i^\epsilon - b_i^\epsilon e^{-\frac{c_i^\epsilon}{\epsilon}(1-x)}}\right), \tag{1.24}$$

for $x \in (0, 1)$, $\epsilon > 0$, $i = 1, 2$. *Here* a_i^ϵ*'s,* b_i^ϵ*'s and* c_i^ϵ*'s are constants satisfying*

$$a_i^\epsilon \to e^{\phi^*/2} + 1,\ b_i^\epsilon \to e^{\phi^*/2} - 1 \text{ and } c_i^\epsilon \to \sqrt{\alpha} \text{ as } \epsilon \downarrow 0,\ \text{for } i = 1, 2$$

where ϕ^* *is defined in (1.23).*

Note that in Theorem 1.3 (ii), we have $\phi^* = \phi_0(1)$ if $\gamma = 0$ and $0 < \phi^* < \phi_0(1)$ if $0 < \gamma < \infty$. Theorem 1.3 (i) shows that there is no boundary layer if $\frac{\eta_\epsilon}{\epsilon} \to \infty$ as $\epsilon \downarrow 0$. Theorem 1.3 (ii) provides the existence of boundary layers having asymptotic behaviors exhibited by (1.24). Consequently, we obtain the boundary estimates of ϕ at $x = -1$ since the solution ϕ is an odd function on $[-1, 1]$. One may refer to Figure 5.1 for the solution profile.

Theorem 1.2 and 1.3 imply $\int_{-1}^{1} e^{\pm\phi} dx \to 2$ as $\epsilon \to 0$. Hence the PB_n equation (1.14) is asymptotically close to the following PB equation:

$$\epsilon^2 \phi_{xx} = \frac{\alpha}{2} e^{\phi} - \frac{\beta}{2} e^{-\phi}, \quad \text{for } x \in (-1, 1), \tag{1.25}$$

i.e., the equation (1.15) with $\alpha_1 = \alpha/2 = \beta/2 = \beta_1$. It would be expected that as $\alpha = \beta$, the solutions of the PB_n equation (1.14) and the PB equation (1.25) have the same asymptotic behavior. Such a result can be stated as follows:

THEOREM 1.4. *Assume* $\alpha = \beta > 0$ *and* $\phi_0(-1) = -\phi_0(1) < 0$. *Under the boundary condition (1.19), let* $\phi, w \in C^\infty((-1, 1)) \cap C^2([-1, 1])$ *be the solution of (1.14) and (1.25), respectively. Then* w *is odd, monotone increasing on* $[-1, 1]$, *convex on* $(0, 1)$, *and concave on* $(-1, 0)$. *Furthermore,*

$$\lim_{\epsilon\downarrow 0} |\phi(x) - w(x)| = 0,\ \forall\ x \in [-1, 1]. \tag{1.26}$$

The numerical results of the maximum norms $\|\phi - w\|_\infty$'s are presented in the second table of Table 5.1 which may support Theorem 1.4. To see the difference between the PB_n equation (1.14) and the PB equation (1.25), we set $\alpha < \beta$ and study the asymptotic behaviors of the solution ϕ of (1.14) and the solution w of (1.25) as ϵ goes to zero. The interior estimates of ϕ are stated as follows:

THEOREM 1.5. *(Interior Estimates) Assume* $0 < \alpha < \beta$. *For* $\epsilon > 0$, *let* $\phi \in C^\infty((-1,1)) \cap C^2([-1,1])$ *be the solution of (1.14) with the boundary condition (1.19). For any compact subset* K *of* $(-1,1)$, *we have*

(i) $\sup_{x,y\in K} |\phi(x) - \phi(y)|$ *exponentially converges to zero as* ϵ *goes to zero.*

(ii) There exists positive constant $C_{K,\alpha,\beta}$ *depending only on* α, β *and* K *such that*

$$\left| \phi(x) - \phi(\pm 1) - \log \frac{1}{\epsilon^2} \right| \leq C_{K,\alpha,\beta}, \tag{1.27}$$

for all $x \in K$.

Theorem 1.5 (ii) presents that the interior value $\phi(x) - \phi(\pm 1)$ tends to infinity with the leading order term $\log \frac{1}{\epsilon^2}$ and the second order term $O(1)$ as ϵ goes zero. This also shows $\lim_{\epsilon\downarrow 0} \frac{\phi(x)-\phi(\pm 1)}{\log \epsilon^{-2}} = 1$ uniformly in any compact subset K of $(-1,1)$.

When $\phi_0(1) = \phi_0(-1)$, the asymptotic behaviors of $\phi(-1)$, $\phi(1)$ are depicted by

$$\lim_{\epsilon\downarrow 0} \phi(1) = \phi_0(1) \qquad \text{if } \lim_{\epsilon\downarrow 0} \frac{\eta_\epsilon}{\epsilon^2} = 0, \tag{1.28}$$

$$\phi(1) = \phi_0(1) + \frac{\gamma(\beta-\alpha)}{2} \qquad \text{if } \eta_\epsilon = \gamma\epsilon^2, \tag{1.29}$$

$$\phi(1) = \phi_0(1) + \frac{\eta_\epsilon(\beta-\alpha)}{2\epsilon^2} \to \infty \quad \text{if } \lim_{\epsilon\downarrow 0} \frac{\eta_\epsilon}{\epsilon^2} = \infty \tag{1.30}$$

where γ is a positive constant independent of ϵ. These formulas can be proved by Remark 3.1 in Section 3. On the other hand, as $\phi_0(1) \neq \phi_0(-1)$, the asymptotic behaviors of $\phi(-1)$, $\phi(1)$ are given by

THEOREM 1.6. *Assume* $0 < \alpha < \beta$ *and* $\phi_0(1) \neq \phi_0(-1)$. *Let* $\phi \in C^\infty((-1,1)) \cap C^2([-1,1])$ *be the solution of the equation (1.14) with the boundary condition (1.19). Then* ϕ *has the following properties at* $x = \pm 1$:

(i) If $\frac{\eta_\epsilon}{\epsilon^2} \to 0$ *as* $\epsilon \downarrow 0$, *then* $\lim_{\epsilon\downarrow 0} \phi(1) = \phi_0(1)$ *and* $\lim_{\epsilon\downarrow 0} \phi(-1) = \phi_0(-1)$ *and*

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \phi'(1) = \frac{(\alpha-\beta)e^{\phi_0(-1)/2}}{e^{\phi_0(1)/2} + e^{\phi_0(-1)/2}}\,, \quad \lim_{\epsilon\downarrow 0} \epsilon^2 \phi'(-1) = \frac{(\beta-\alpha)e^{\phi_0(1)/2}}{e^{\phi_0(1)/2} + e^{\phi_0(-1)/2}}\,; \tag{1.31}$$

(ii) If $\eta_\epsilon = \gamma\epsilon^2$ *for some positive constant* γ *independent of* ϵ, *then* $\lim_{\epsilon\downarrow 0}(\phi(1), \phi(-1)) = (\phi_1^*, \phi_2^*)$ *satisfies* $\lim_{\epsilon\downarrow 0} \epsilon^2 \phi'(1) = \frac{\phi_0(1)-\phi_1^*}{\gamma}$, $\lim_{\epsilon\downarrow 0} \epsilon^2 \phi'(-1) = -\frac{\phi_0(-1)-\phi_2^*}{\gamma}$ *and*

$$\begin{cases} \phi_1^* + \phi_2^* = \phi_0(1) + \phi_0(-1) + \gamma(\beta-\alpha), \\ (\phi_1^* - \phi_0(1))e^{\phi_1^*/2} - (\phi_2^* - \phi_0(-1))e^{\phi_2^*/2} = 0, \\ \phi_1^* > \phi_0(1),\ \phi_2^* > \phi_0(-1)\,; \end{cases} \tag{1.32}$$

(iii) If $\frac{\eta_\epsilon}{\epsilon^2} \to \infty$ *as* $\epsilon \downarrow 0$, *then* $\lim_{\epsilon\downarrow 0}(\phi(-1) - \phi(1)) = 0$ *and*

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \phi'(-1) = -\lim_{\epsilon\downarrow 0} \epsilon^2 \phi'(1) = -\frac{\alpha-\beta}{2}. \tag{1.33}$$

As $\phi_0(1) = \phi_0(-1)$, the boundary estimates of ϕ are given by

THEOREM 1.7. *(Boundary Estimates) Under the same hypotheses of Theorem 1.5, if $\phi_0(1) = \phi_0(-1)$, then*

$$\phi_1^\epsilon(x) \le \phi(x) - \phi(1) - \log\frac{(\alpha-\beta)^2}{2\epsilon^2} \le \phi_2^\epsilon(x),$$

$$\phi_1^\epsilon(x) = 2\log\frac{1 - d_{1,1}^\epsilon e^{-\frac{c_{1,1}^\epsilon}{\epsilon}(1-x)}}{\sqrt{\alpha+\beta}\left(1 + d_{1,2}^\epsilon e^{-\frac{c_{1,1}^\epsilon}{\epsilon}(1-x)}\right)}, \quad \phi_2^\epsilon(x) = 2\log\frac{1 - d_{2,1}^\epsilon e^{-\frac{c_{2,2}^\epsilon}{\epsilon}(1-x)}}{1 + d_{2,2}^\epsilon e^{-\frac{c_{2,2}^\epsilon}{\epsilon}(1-x)}},$$

for $\epsilon > 0$ sufficiently small and $x \in (y_\epsilon, 1)$, where $0 < y_\epsilon < 1$ is a constant sufficiently close to 1 and $c_{i,j}^\epsilon$'s, $d_{i,j}^\epsilon$'s $(i, j = 1, 2)$ are constants satisfying $c_{1,1}^\epsilon \to \sqrt{2(\alpha+\beta)}$, $c_{2,2}^\epsilon \to \sqrt{\frac{3}{2}}(\alpha\beta)^{1/4}$, $d_{1,1}^\epsilon$, $d_{1,2}^\epsilon$, and $d_{2,1}^\epsilon \to 1$ and $d_{2,2}^\epsilon \to 2+\sqrt{3}$ as $\epsilon \downarrow 0$.

Similar results also hold for $\phi_0(1) \neq \phi_0(-1)$. One may refer to Figure 5.2-5.4 for the solution profile. To prove Theorem 1.1-1.7, we need the following results:

$$\frac{\epsilon^2}{2}(\phi'(x))^2 = \frac{\alpha}{\int_{-1}^1 e^{\phi(y)}dy}e^{\phi(x)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)}dy}e^{-\phi(x)} + C_\epsilon\,, \tag{1.34}$$

$$\epsilon^2\phi'''(x)\phi'(x) = \left(\frac{\alpha}{\int_{-1}^1 e^{\phi(y)}dy}e^{\phi(x)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)}dy}e^{-\phi(x)}\right)(\phi'(x))^2\,, \tag{1.35}$$

$$\begin{aligned}&\epsilon^2(\phi''(a)\phi'(a) - \phi''(0)\phi'(0))\\ &\ge \int_0^a \left(\frac{\alpha}{\int_{-1}^1 e^{\phi(y)}dy}e^{\phi(x)} + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)}dy}e^{-\phi(x)}\right)(\phi'(x))^2dx,\ \forall\, a \in (0,1]\,,\end{aligned} \tag{1.36}$$

and

$$\begin{aligned}&\frac{\epsilon^2}{2}\left[(\phi'(1))^2 + (\phi'(-1))^2 - \int_{-1}^1(\phi'(x))^2dx\right]\\ =&\frac{\alpha}{\int_{-1}^1 e^{\phi(y)}dy}\left(e^{\phi(1)} + e^{\phi(-1)}\right) + \frac{\beta}{\int_{-1}^1 e^{-\phi(y)}dy}\left(e^{-\phi(1)} + e^{-\phi(-1)}\right) - (\alpha+\beta)\,,\end{aligned} \tag{1.37}$$

for $|x| < 1$ where C_ϵ is a constant. The proof of (1.34)-(1.37) are quite standard so we omit the detail here.

1.2. Organization of the paper. The rest of this paper is organized as follows: The proof of the electroneutral cases ($\alpha = \beta$), Theorem 1.2 and 1.3 are showed in Section 2. In Section 3, we prove the non-electroneutral cases, Theorem 1.5-1.7. Theorem 1.4 and 4.2 related to the difference/comparison between the PB_n equation (1.14) and the PB equation (1.25) are given in Section 4. In Section 5, the numerical computations to solve the PB_n equation (1.14) and the PB equation (1.15) are performed using convex iteration method based on finite element discretization [39]. Finally, the proof of Theorem 1.1 is given in Section 6.

2. Electroneutral cases: Proof of Theorem 1.2 and 1.3.

2.1. Proof of Theorem 1.2. Let $\psi(x) = -\phi(-x)$ for $x \in [-1,1]$. Since $\alpha = \beta$ and $\phi_0(1) = -\phi_0(-1) > 0$, it is obvious that ψ is also a solution of the equation (1.14) with the boundary condition (1.19). Hence by Theorem 1.1, $\psi \equiv \phi$ i.e., ϕ is an odd function on $[-1,1]$. Consequently,

$$\int_{-1}^{1} e^{\phi(y)}dy = \int_{-1}^{1} e^{-\phi(y)}dy \quad \text{and} \quad \phi''(0) = \phi(0) = 0\,. \tag{2.1}$$

Moreover, (2.1), and $\alpha = \beta$ may transform (1.14) and (1.36) into

$$\epsilon^2\phi''(x)\phi(x) = \frac{\alpha}{\int_{-1}^{1} e^{\phi(y)}dy}\left(e^{\phi(x)} - e^{-\phi(x)}\right)\phi(x) \geq 0,\ \forall\, x \in (-1,1)\,, \tag{2.2}$$

and

$$\epsilon^2\phi''(x)\phi'(x) \geq \frac{\alpha}{\int_{-1}^{1} e^{\phi(y)}dy}\int_0^x \left(e^{\phi(y)} + e^{-\phi(y)}\right)(\phi'(y))^2dy,\, \forall\, x \in [0,1]\,, \tag{2.3}$$

which imply that ϕ'', ϕ' and ϕ do not change sign and share the same sign on $[0,1]$. In particular, we have $\phi(1)\phi'(1) \geq 0$. By the boundary condition (1.19) and $\eta_\epsilon \geq 0$, we get $0 < \phi(1), \eta_\epsilon\phi'(1) \leq \phi_0(1)$. So ϕ, ϕ' and ϕ'' are non-negative on $(0,1]$ i.e., ϕ is convex and monotonically increasing on $(0,1]$. On the other hand, since ϕ is odd on $[-1,1]$, then ϕ is concave and monotonically increasing on $[-1,0)$. Thus $\max_{x\in[-1,1]}|\phi(x)| \leq \phi_0(1)$ and (2.2) implies $\frac{\epsilon^2}{2}(\phi^2(x))'' \geq \frac{\alpha}{e^{\phi_0(1)}}\phi^2(x)$ for $x \in (-1,1)$. By the standard comparison theorem, we have $\phi^2(x) \leq \phi_0^2(1)\left(e^{-\frac{\widetilde{M}}{\epsilon}(1+x)} + e^{-\frac{\widetilde{M}}{\epsilon}(1-x)}\right)$ for $x \in [-1,1]$ where $\widetilde{M} = \sqrt{\frac{2\alpha}{e^{\phi_0(1)}}}$. Therefore, we complete the proof of Theorem 1.2.

2.2. Proof of Theorem 1.3. In order to prove Theorem 1.3, we need the following lemma:

LEMMA 2.1. *Assume $\alpha = \beta > 0$ and $-\phi_0(-1) = \phi_0(1) > 0$. Let $\phi \in C^\infty((-1,1)) \cap C^2([-1,1])$ satisfy (1.14) and (1.19). Then $\int_{-1}^{1} e^{\phi(y)}dy = \int_{-1}^{1} e^{-\phi(y)}dy \to 2$ and $C_\epsilon \to -\alpha$ as $\epsilon \downarrow 0$, where C_ϵ is defined in (1.34).*

Proof. By (1.21) and the dominated convergence theorem, we obtain

$$\lim_{\epsilon\downarrow 0}\int_{-1}^{1} e^{\phi(y)}dy = \lim_{\epsilon\downarrow 0}\int_{-1}^{1} e^{-\phi(y)}dy = 2. \tag{2.4}$$

Now we claim that $C_\epsilon \to -\alpha$ as $\epsilon \downarrow 0$. Since $\alpha = \beta$ and $\int_{-1}^{1} e^{\phi(y)}dy = \int_{-1}^{1} e^{-\phi(y)}dy$, then (1.34) becomes

$$\frac{\epsilon^2}{2}(\phi'(x))^2 = \frac{\alpha}{\int_{-1}^{1} e^{\phi(y)}dy}\left(e^{\phi(x)} + e^{-\phi(x)}\right) + C_\epsilon \quad \text{for} \quad |x| < 1\,. \tag{2.5}$$

By Theorem 1.2, we have $-\phi_0(1) \leq \phi(-1) \leq \phi(1) \leq \phi_0(1)$. Hence there exists $x_* \in (-1,1)$ such that $0 \leq \phi'(x_*) = [\phi(1) - \phi(-1)]/2 \leq \phi_0(1)$. Setting $x = 0$ and $x = x_*$ in (2.5) and using $\phi(0) = 0$ (see (2.1)) and $e^{\phi(x_*)} + e^{-\phi(x_*)} \geq 2$, we obtain

$$-\frac{2\alpha}{\int_{-1}^{1} e^{\phi(y)}dy} \leq C_\epsilon \leq \frac{\epsilon^2}{2}\phi_0^2(1) - \frac{2\alpha}{\int_{-1}^{1} e^{\phi(y)}dy}. \tag{2.6}$$

By (2.4) and (2.6), we get $C_\epsilon \to -\alpha$ as $\epsilon \downarrow 0$. □

Now we state the proof of Theorem 1.3. By (2.5) and (2.6), it is easy to get

$$0 \le \epsilon^2(\phi'(x))^2 - \frac{2\alpha}{\int_{-1}^1 e^{\phi(y)}dy}\left(e^{\phi(x)/2} - e^{-\phi(x)/2}\right)^2 \le \epsilon^2\phi_0^2(1)\,. \tag{2.7}$$

By Theorem 1.2, ϕ is positive and monotonically increasing in $(0, 1]$. Hence

$$\begin{aligned}&\sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)}dy}}\left(e^{\phi(x)/2} - e^{-\phi(x)/2}\right)\\ &\le \epsilon\phi'(x) \le \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)}dy}}\left(e^{\phi(x)/2} - e^{-\phi(x)/2}\right) + \epsilon\phi_0(1), \quad \text{for} \quad x \in (0, 1]\,.\end{aligned} \tag{2.8}$$

Setting $x = 1$ in (2.8) and using the boundary condition (1.19), we get

$$\begin{aligned}&\sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)}dy}}(e^{\phi(1)/2} - e^{-\phi(1)/2}) \le \frac{\epsilon}{\eta_\epsilon}(\phi_0(1) - \phi(1))\\ &= \epsilon\phi'(1) \le \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)}dy}}\left(e^{\phi(1)/2} - e^{-\phi(1)/2}\right) + \epsilon\phi_0(1).\end{aligned} \tag{2.9}$$

Now we deal with (2.9) in following three cases:
Case (i): $\frac{\eta_\epsilon}{\epsilon} \to \infty$ as $\epsilon \downarrow 0$.

We remark that $0 < \phi(1) \le \phi_0(1)$. Then (2.9) gives $\sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)}dy}}\left(e^{\phi(1)/2} - 1\right) \le \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)}dy}}\left(e^{\phi(1)/2} - e^{-\phi(1)/2}\right) \le \frac{\epsilon}{\eta_\epsilon}(\phi_0(1) - \phi(1)) \le \frac{\epsilon}{\eta_\epsilon}\phi_0(1)$, i.e., $0 < \phi(1) \le 2\log\left(1 + \frac{\epsilon}{\eta_\epsilon}\phi_0(1)\sqrt{\frac{\int_{-1}^1 e^{\phi(y)}dy}{2\alpha}}\right)$. Since ϕ is a monotone increasing and odd function on $[-1, 1]$, we have $\max_{x\in[-1,1]}|\phi(x)| \le 2\log\left(1 + \frac{\epsilon}{\eta_\epsilon}\phi_0(1)\sqrt{\frac{\int_{-1}^1 e^{\phi(y)}dy}{2\alpha}}\right)$ which and (2.4) imply $\lim_{\epsilon\downarrow 0}|\phi(x)| = 0$ uniformly in $[-1, 1]$. Then the boundary condition (1.19) gives $\lim_{\epsilon\downarrow 0}\eta_\epsilon\phi'(1) = \phi_0(1)$ so we complete the proof of Theorem 1.3 (i).
Case (ii): $\frac{\eta_\epsilon}{\epsilon} \to \gamma$ as $\epsilon \downarrow 0$, for some positive constant γ.

Due to $|\phi(1)| \le \phi_0(1)$, we assume $\limsup_{\epsilon\downarrow 0}\phi(1) = \phi_s^*$ and $\liminf_{\epsilon\downarrow 0}\phi(1) = \phi_i^*$. Then (2.9) implies ϕ_s^* and ϕ_i^* are the roots of (1.23) with $0 < \phi_i^* \le \phi_s^* < \phi_0(1)$. It is obvious that the equation (1.23) has only one root in the interval $(0, \phi_0(1))$. Consequently, $\phi_s^* = \phi_i^* \equiv \phi^*$ and $\lim_{\epsilon\downarrow 0}\phi(1) = \phi^* \in (0, \phi_0(1))$. Therefore, by (2.9), we obtain (1.22) and complete the proof of Theorem 1.3 (ii) for the case of $\gamma > 0$.
Case (iii): $\frac{\eta_\epsilon}{\epsilon} \to 0$ as $\epsilon \downarrow 0$.

For convenience, let $C(\epsilon, \alpha, \phi(1)) = \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)}dy}}\left(e^{\phi(1)/2} - e^{-\phi(1)/2}\right)$. By Theorem 1.2, there exists a positive constant C_1 independent of ϵ such that $0 \le C(\epsilon, \alpha, \phi(1)) \le C_1$. Thus by (2.9), we have

$$C(\epsilon, \alpha, \phi(1)) \le \frac{\epsilon}{\eta_\epsilon}(\phi_0(1) - \phi(1)) = \epsilon\phi'(1) \le C(\epsilon, \alpha, \phi(1)) + \epsilon\phi_0(1), \tag{2.10}$$

for $\epsilon \in (0, 1)$, and then

$$(1 - \eta_\epsilon)\phi_0(1) - \frac{\eta_\epsilon}{\epsilon}C(\epsilon, \alpha, \phi(1)) \le \phi(1) \le \phi_0(1) - \frac{\eta_\epsilon}{\epsilon}C(\epsilon, \alpha, \phi(1)). \tag{2.11}$$

Note that $\eta_\epsilon \to 0$ as $\epsilon \to 0$ and $\phi_0(1)$ is independent of ϵ. Hence (2.11) implies $\lim_{\epsilon\downarrow 0}\phi(1) = \phi_0(1)$, and

$$\lim_{\epsilon\downarrow 0} C(\epsilon,\alpha,\phi(1)) = \sqrt{\alpha}\left(e^{\phi_0(1)/2} - e^{-\phi_0(1)/2}\right). \tag{2.12}$$

Here we have used the assumption $\frac{\eta_\epsilon}{\epsilon} \to 0$ i.e., $\frac{\epsilon}{\eta_\epsilon} \to \infty$ as $\epsilon \to 0$. As a consequence, by (2.10) and (2.12), we obtain $\lim_{\epsilon\downarrow 0}\epsilon\phi'(1) = \sqrt{\alpha}\left(e^{\phi_0(1)/2} - e^{-\phi_0(1)/2}\right)$ and complete the proof of Theorem 1.3 (ii) for the case of $\gamma = 0$.

Now we want to prove (1.24). Let $\delta_\epsilon(\alpha) = \sqrt{\frac{2\alpha}{\int_{-1}^1 e^{\phi(y)}dy}}$. Then Lemma 2.1 gives

$$\lim_{\epsilon\to 0}\delta_\epsilon(\alpha) = \sqrt{\alpha}\,. \tag{2.13}$$

Using (2.8) and $\phi > 0$ in $(0,1]$, we have

$$\frac{\phi'(z)}{e^{\phi(z)/2} - e^{-\phi(z)/2} + \frac{\varepsilon\phi_0(1)}{\delta_\epsilon(\alpha)}} \le \frac{\delta_\epsilon(\alpha)}{\epsilon} \le \frac{\phi'(z)}{e^{\phi(z)/2} - e^{-\phi(z)/2}}\,, \quad \forall\, z \in (0,1]. \tag{2.14}$$

For $x \in (0,1)$, we may integrate (2.14) over $(x,1)$ to get

$$\int_x^1 \frac{\phi'(z)dz}{e^{\phi(z)/2} - e^{-\phi(z)/2} + \frac{\varepsilon\phi_0(1)}{\delta_\epsilon(\alpha)}} = \frac{2}{t_+ - t_-}\log\frac{(e^{\phi(1)/2} - t_+)(e^{\phi(x)/2} - t_-)}{(e^{\phi(1)/2} - t_-)(e^{\phi(x)/2} - t_+)} \tag{2.15}$$

where $t_\pm = \frac{1}{2}\left(-\frac{\varepsilon\phi_0(1)}{\delta_\epsilon(\alpha)} \pm \sqrt{\left(\frac{\varepsilon\phi_0(1)}{\delta_\epsilon(\alpha)}\right)^2 + 4}\right)$. Similarly, we may also have

$$\int_x^1 \frac{\phi'(z)dz}{e^{\phi(z)/2} - e^{-\phi(z)/2}} = \log\frac{(e^{\phi(1)/2} - 1)(e^{\phi(x)/2} + 1)}{(e^{\phi(1)/2} + 1)(e^{\phi(x)/2} - 1)}\,. \tag{2.16}$$

Hence (2.14)-(2.16) imply

$$\begin{aligned} &2\log\frac{t_+(e^{t_\epsilon/2} - t_-) - t_-(e^{t_\epsilon/2} - t_+)e^{-(t_+ - t_-)\delta_\epsilon(\alpha)(1-x)/2\epsilon}}{(e^{t_\epsilon/2} - t_-) - (e^{t_\epsilon/2} - t_+)e^{-(t_+ - t_-)\delta_\epsilon(\alpha)(1-x)/2\epsilon}} \\ &\le \phi(x) \le 2\log\frac{(e^{t_\epsilon/2} + 1) + (e^{t_\epsilon/2} - 1)e^{-\delta_\epsilon(\alpha)\cdot(1-x)/\epsilon}}{(e^{t_\epsilon/2} + 1) - (e^{t_\epsilon/2} - 1)e^{-\delta_\epsilon(\alpha)(1-x)/\epsilon}} \quad \forall\, x \in (0,1) \end{aligned} \tag{2.17}$$

where $t_\epsilon = \phi(1)$. Note that (2.13) shows $t_\pm \to \pm 1$ and $\delta_\epsilon(\alpha) \to \sqrt{\alpha}$ as $\epsilon \downarrow 0$. Thus (1.22) and (2.17) give (1.24) provided $\frac{\eta_\epsilon}{\epsilon} \to \gamma$ as $\epsilon \downarrow 0$, where γ is a nonnegative constant. Therefore, we complete the proof of Theorem 1.3.

3. Non-electroneutral cases: Proof of Theorem 1.5 – 1.7. In this section, we assume that $0 < \alpha < \beta$ in the whole section. To prove Theorem 1.5 – 1.7, we need the following lemma:

Lemma 3.1. *Let* $B(\phi)(x) = \frac{\alpha}{\int_{-1}^1 e^{\phi(y)}dy}e^{\phi(x)} - \frac{\beta}{\int_{-1}^1 e^{-\phi(y)}dy}e^{-\phi(x)}$ *for* $x \in [-1,1]$ *where* ϕ *is the solution of (1.14) and (1.19). Then*

(i) $\int_{-1}^1 B(\phi)(x)dx = \alpha - \beta < 0$*;*

(ii) $B(\phi)(x)$ *is monotonically increasing to* $\phi(x)$ *in the sense that if* $\phi(x) < \phi(z)$*, then* $B(\phi)(x) < B(\phi)(z)$*;*

(iii) $B(\phi)(x)$ *has no negative interior minima and positive interior maxima.*

(iv) Assume that

$$\sup_{\epsilon>0}\int_{-1}^{1} e^{\phi(y)}dy \int_{-1}^{1} e^{-\phi(y)}dy < \infty\,. \tag{3.1}$$

Then there exists $C_5 > 0$ independent of ϵ such that $\forall x \in [-1,1]$,

$$B(\phi)(x) \geq \min\{B(\phi)(1), B(\phi)(-1)\}\left(e^{-\frac{C_5(1+x)}{\epsilon}} + e^{-\frac{C_5(1-x)}{\epsilon}}\right). \tag{3.2}$$

Proof. It is easy to get (i) and (ii) immediately from the definition of $B(\phi)(x)$. Since ϕ solves (1.14), then $B(\phi)(x)$ satisfies

$$\epsilon^2 B(\phi)''(x) = \left(\frac{\alpha}{\int_{-1}^{1} e^{\phi(y)}dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^{1} e^{-\phi(y)}dy} e^{-\phi(x)} + \epsilon^2 \phi'^2(x)\right) B(\phi)(x)\,, \tag{3.3}$$

for $x \in (-1,1)$. Thus (iii) is obtained by (3.3) and the standard maximum principle. Now we want to prove (iv). Note that (i)-(iii) give $\min\{B(\phi)(1), B(\phi)(-1)\} < 0$. On the other hand, by (3.1), there exists $C_5 > 0$ independent of ϵ such that $\frac{\alpha}{\int_{-1}^{1} e^{\phi(y)}dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^{1} e^{-\phi(y)}dy} e^{-\phi(x)} + \epsilon^2\phi'^2(x) \geq \frac{2\sqrt{\alpha\beta}}{\sqrt{\int_{-1}^{1} e^{\phi(y)}dy \int_{-1}^{1} e^{-\phi(y)}dy}} \geq C_5^2$. Next we define a function v by $v(x) = \min\{B(\phi)(1), B(\phi)(-1)\}\left(e^{-\frac{C_5(1+x)}{\epsilon}} + e^{-\frac{C_5(1-x)}{\epsilon}}\right)$ for $x \in [-1,1]$. Then $v(x) < 0$ for $x \in [-1,1]$, $v(\pm 1) \leq \min\{B(\phi)(1), B(\phi)(-1)\} \leq B(\phi)(\pm 1)$ and $\epsilon^2 v''(x) - \left(\frac{\alpha}{\int_{-1}^{1} e^{\phi(y)}dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^{1} e^{-\phi(y)}dy} e^{-\phi(x)} + \epsilon^2\phi'^2(x)\right) v(x) \geq 0$ for $x \in (-1,1)$. By the standard comparison theorem, $v(x) \leq B(\phi)(x)$ for all $x \in (-1,1)$. Therefore, we obtain (3.2) and complete the proof of Lemma 3.1. □

The interior gradient estimate of ϕ is crucial for the proof of Theorem 1.5 and 1.7. We state the result as follows:

THEOREM 3.2. *Assume $0 < \alpha < \beta$. Let $\phi \in C^\infty((-1,1)) \cap C^2([-1,1])$ be the solution of (1.14) and (1.19). Then*
(i) if $\phi_0(-1) = \phi_0(1)$, then ϕ is even, concave on $(-1,1)$, increasing on $(-1,0)$, and decreasing on $(0,1)$,
(ii) if $\phi_0(-1) \neq \phi_0(1)$, then ϕ is concave on $(-1,1)$ and there exists $x_\epsilon \in (-1,1)$ such that ϕ is increasing on $(-1,x_\epsilon)$ and decreasing on $(x_\epsilon,1)$.
(iii)

$$|\phi'(x)| \leq \max\{|\phi'(-1)|, |\phi'(1)|\}\left(e^{-\frac{\sqrt{2\alpha}}{2\epsilon}(1+x)} + e^{-\frac{\sqrt{2\alpha}}{2\epsilon}(1-x)}\right), \forall\, x \in (-1,1)\,. \tag{3.4}$$

Proof. Firstly, we prove (i). Suppose $\phi_0(-1) = \phi_0(1)$. Let $\psi(x) = \phi(-x)$ for $x \in [-1,1]$. Then it is obvious that ψ also satisfies (1.14) and (1.19). By Theorem 1.1, $\phi \equiv \psi$ i.e., ϕ is an even function on $[-1,1]$. Hence $\phi'(0) = 0$, along with (1.36), we have

$$\phi''(x)\phi'(x) \geq 0\,, \quad \forall\, x \in (0,1)\,, \tag{3.5}$$

which implies that both ϕ'' and ϕ' never change sign on $(0,1)$. On the other hand, since ϕ is even on $[-1,1]$, then $\phi'(x) = -\phi'(-x)$ and $\phi''(x) = \phi''(-x)$ for $x \in (0,1)$. Thus the sign of ϕ'' on $(-1,0)$ is exactly same as the sign of ϕ'' on $(0,1)$ i.e., ϕ'' never changes sign on $(-1,1)$. By Lemma 3.1(i) and (1.14), we have that $\phi''(x_0) < 0$ for some $x_0 \in (-1,1)$. Consequently, $\phi'' \leq 0$ on $(-1,1)$ and ϕ is concave on $(-1,1)$. Moreover, (3.5) and $\phi'(x) = -\phi'(-x)$ give $\phi'(x) \leq 0$ for $x \in (0,1)$ and $\phi'(x) \geq 0$ for $x \in (-1,0)$. Therefore, we complete the proof of Theorem 3.2 (i).

For the case of $\phi_0(-1) \neq \phi_0(1)$, without loss of generality, we assume that

$$\phi_0(-1) < \phi_0(1)\,. \tag{3.6}$$

Claim 1. There exists $x_\epsilon \in (-1,1)$ such that $\phi'(x_\epsilon) = 0$.

Proof. We state the proof by contradiction. Suppose $\phi'(x) < 0$ for $x \in (-1,1)$. Then by (1.19) and (3.6), we have $\phi(-1) \geq \phi(1) = \phi_0(1) - \eta_\epsilon \phi'(1) \geq \phi_0(1) > \phi_0(-1) \geq \phi_0(-1) + \eta_\epsilon \phi'(-1) = \phi(-1)$, which gives a contradiction. On the other hand, suppose $\phi'(x) > 0$ for $x \in (-1,1)$. Then by (1.19), we have

$$\phi_0(-1) \leq \phi(-1) \leq \phi(x) \leq \phi(1) \leq \phi_0(1),\ \forall\, x \in [-1,1]. \tag{3.7}$$

Hence ϕ satisfies the condition (3.1). Since $\phi(-1) \leq \phi(1)$, then Lemma 3.1(ii) implies $B(\phi)(-1) \leq B(\phi)(1)$. Furthermore, Lemma 3.1(i) and (3.2) give

$$\frac{2\epsilon}{C_5}\,(B(\phi)(-1))\left(1 - e^{-2C_5/\epsilon}\right) = (B(\phi)(-1)) \int_{-1}^{1} \left(e^{-\frac{C_5(1+x)}{\epsilon}} + e^{-\frac{C_5(1-x)}{\epsilon}}\right) dx \leq \alpha - \beta. \tag{3.8}$$

Note that (3.7) implies $|B(\phi)(-1)| = O(1)$. Letting $\epsilon \downarrow 0$ on (3.8), we get $\alpha \geq \beta$ which contradicts to the hypothesis $\alpha < \beta$. Therefore, we complete the proof of Claim 1. □

As for (1.36), we integrate (1.35) over (x_ϵ, x) so we have $\phi''(x)\phi'(x) > 0$ for $x \in (x_\epsilon, 1)$. Similarly, we obtain $\phi''(x)\phi'(x) < 0$ for $x \in (-1, x_\epsilon)$. Here we have used the fact that the solution ϕ is nontrivial due to the boundary condition (1.19) and the assumption (3.6). Moreover, the proof of Claim 1 also gives $\phi'(x_1)\phi'(x_2) < 0$ for any $x_1 \in (x_\epsilon, 1)$ and $x_2 \in (-1, x_\epsilon)$. Consequently, ϕ'' never changes sign on $(-1,1)$. Then by Lemma 3.1(i), we obtain $\phi''(x) \leq 0$ for $x \in (-1,1)$, $\phi'(x) < 0$ for $x \in (x_\epsilon, 1)$ and $\phi'(x) > 0$ for $x \in (-1, x_\epsilon)$. Therefore, we complete the proof of Theorem 3.2 (ii).

Now we want to prove (3.4). Firstly, we state a crucial estimate as follows:

Claim 2.

$$4 \leq \int_{-1}^{1} e^{\phi(y)} dy \int_{-1}^{1} e^{-\phi(y)} dy \leq \frac{4\beta}{\alpha}. \tag{3.9}$$

Proof. The left side of (3.9) is trivial due to the application of Hölder's inequality. From the proof of Theorem 3.2 (i) and (ii), we have $\phi''(x) \leq 0$ for $x \in (-1,1)$. Hence by (1.14), we obtain $\frac{\alpha e^{\phi(x)}}{\int_{-1}^{1} e^{\phi(y)} dy} - \frac{\beta e^{-\phi(x)}}{\int_{-1}^{1} e^{-\phi(y)} dy} = \epsilon^2 \phi''(x) \leq 0$ for $x \in (-1,1)$ which implies $e^{2\phi(x)} \leq \frac{\beta \int_{-1}^{1} e^{\phi(y)} dy}{\alpha \int_{-1}^{1} e^{-\phi(y)} dy}$ for $x \in (-1,1)$. Integrating this inequality over $(-1,1)$ and using Hölder's inequality, we get $\frac{1}{2}\left(\int_{-1}^{1} e^{\phi(x)} dx\right)^2 \leq \int_{-1}^{1} e^{2\phi(x)} dx \leq \frac{2\beta \int_{-1}^{1} e^{\phi(y)} dy}{\alpha \int_{-1}^{1} e^{-\phi(y)} dy}$, which gives the right side of (3.9) and complete the proof of Claim 2. □

By (1.14), (3.9) and (1.34), it is easy to calculate that

$$\frac{\epsilon^2}{2}(\phi'^2)'' = \epsilon^2(\phi'')^2 + \left(\frac{\alpha}{\int_{-1}^{1} e^{\phi(y)} dy} e^{\phi(x)} + \frac{\beta}{\int_{-1}^{1} e^{-\phi(y)} dy} e^{-\phi(x)}\right) \phi'^2 \tag{3.10}$$

$$\geq 2\left(\frac{\alpha\beta}{\int_{-1}^{1} e^{\phi(y)}dy\int_{-1}^{1} e^{-\phi(y)}dy}\right)^{1/2} \phi'^2 \geq \alpha\,\phi'^2 \quad \text{in } (-1,1)\,.$$

Let $v(x) = \max\{(\phi'(-1))^2, (\phi'(1))^2\}\left(e^{-\frac{\sqrt{2\alpha}}{\epsilon}(1+x)} + e^{-\frac{\sqrt{2\alpha}}{\epsilon}(1-x)}\right)$ for $x \in [-1,1]$. Then $v(\pm 1) \geq (\phi'(\pm 1))^2$ and $\frac{\epsilon^2}{2}v''(x) = \alpha\, v(x)$. Therefore, by (3.10) and the standard comparison theorem, $(\phi'(x))^2 \leq v(x)$ for $x \in (-1,1)$ which implies (3.4) and complete the proof of Theorem 3.2. □

Remark 3.1. *(i) By (1.19) and Theorem 3.2,* $\phi(1) \geq \phi_0(1)$ *and* $\phi(-1) \geq \phi_0(-1)$. *(ii) By (1.14) and Lemma 3.1 (i), we have* $\epsilon^2\int_{-1}^{1}\phi''(x)dx = \int_{-1}^{1} B(\phi)(x)dx = \alpha - \beta$, *i.e.,*

$$\phi'(1) - \phi'(-1) = \frac{\alpha-\beta}{\epsilon^2}. \tag{3.11}$$

Suppose that $\phi_0(1) = \phi_0(-1)$. Then ϕ is even i.e., $\phi(x) = \phi(-x)$ for $x \in [-1,1]$ which implies $\phi'(1) = -\phi'(-1)$. Hence by (3.11), $\phi'(1) = -\phi'(-1) = \frac{\alpha-\beta}{2\epsilon^2}$. Furthermore, by (1.19), we obtain (1.28)-(1.30). However, as $\phi_0(1) \neq \phi_0(-1)$, the asymptotic behavior of $\phi(1)$ is not obvious. Now we state results of this case as follows:

Lemma 3.3. *Assume that* $0 < \alpha < \beta$ *and* $\phi_0(1) \neq \phi_0(-1)$. *Let* $\phi \in C^\infty((-1,1)) \cap C^2([-1,1])$ *be the solution of (1.14) and (1.19). Then there exist* $0 < C_6 \leq C_7 \leq \beta - \alpha$ *independent of* ϵ *such that* $\frac{C_6}{\epsilon^2} \leq \phi'(-1), -\phi'(1) \leq \frac{C_7}{\epsilon^2}$ *as* $0 < \epsilon << 1$.

Proof. From Theorem 3.2 (ii), we have $\phi'(-1), -\phi'(1) \geq 0$. Without loss of generality, we assume that $0 \leq -\phi'(1) \leq \phi'(-1)$. Then by (3.11), we obtain

$$\frac{\beta-\alpha}{2\epsilon^2} \leq \phi'(-1) \leq \frac{\beta-\alpha}{\epsilon^2}. \tag{3.12}$$

Setting $x = 1, -1, x_\epsilon$ in (1.34) and using $\phi'(x_\epsilon) = 0$ (see Claim 1), we derive that

$$\frac{\epsilon^2}{2}(\phi'(1))^2 = \frac{\alpha\left(e^{\phi(1)} - e^{\phi(x_\epsilon)}\right)}{\int_{-1}^{1} e^{\phi(y)}dy} + \frac{\beta\left(e^{-\phi(1)} - e^{-\phi(x_\epsilon)}\right)}{\int_{-1}^{1} e^{-\phi(y)}dy}, \tag{3.13}$$

$$\frac{\epsilon^2}{2}(\phi'(-1))^2 = \frac{\alpha\left(e^{\phi(-1)} - e^{\phi(x_\epsilon)}\right)}{\int_{-1}^{1} e^{\phi(y)}dy} + \frac{\beta\left(e^{-\phi(-1)} - e^{-\phi(x_\epsilon)}\right)}{\int_{-1}^{1} e^{-\phi(y)}dy}. \tag{3.14}$$

By (1.37), (3.13) and (3.14), we have

$$\frac{\alpha}{\int_{-1}^{1} e^{\phi(y)}dy}e^{\phi(x_\epsilon)} + \frac{\beta}{\int_{-1}^{1} e^{-\phi(y)}dy}e^{-\phi(x_\epsilon)} + \frac{\epsilon^2}{4}\int_{-1}^{1}(\phi'(x))^2dx = \frac{\alpha+\beta}{2}. \tag{3.15}$$

Note that by Theorem 3.2 (ii),

$$\phi(\pm 1) \leq \phi(x_\epsilon)\,. \tag{3.16}$$

Thus (3.13)-(3.16) give

$$-\frac{\alpha+\beta}{2} + \frac{\beta}{\int_{-1}^{1} e^{-\phi(y)}dy}e^{-\phi(1)} \leq \frac{\epsilon^2}{2}(\phi'(1))^2 \leq \frac{\beta}{\int_{-1}^{1} e^{-\phi(y)}dy}e^{-\phi(1)}, \tag{3.17}$$

$$-\frac{\alpha+\beta}{2}+\frac{\beta}{\int_{-1}^{1} e^{-\phi(y)}dy}e^{-\phi(-1)} \le \frac{\epsilon^2}{2}(\phi'(-1))^2 \le \frac{\beta}{\int_{-1}^{1} e^{-\phi(y)}dy}e^{-\phi(-1)}\,. \tag{3.18}$$

Moreover, (3.12) and (3.18) imply

$$\frac{(\beta-\alpha)^2}{8\epsilon^2}e^{\phi(-1)} \le \frac{\beta}{\int_{-1}^{1} e^{-\phi(y)}dy} \le \left[\frac{(\beta-\alpha)^2}{2\epsilon^2}+\frac{\alpha+\beta}{2}\right]e^{\phi(-1)}. \tag{3.19}$$

Combining (3.17) and (3.19), we obtain

$$\frac{\epsilon^2}{2}(\phi'(1))^2 \ge -\frac{\alpha+\beta}{2}+\frac{(\beta-\alpha)^2}{8\epsilon^2}e^{\phi(-1)-\phi(1)}\,. \tag{3.20}$$

By (1.19) and the assumption $0 \le -\phi'(1) \le \phi'(-1)$, it is trivial that $\phi(-1)-\phi(1) = \phi_0(-1)-\phi_0(1)+\eta_\epsilon(\phi'(-1)+\phi'(1)) \ge \phi_0(-1)-\phi_0(1)$. Hence by (3.20), there exists $\epsilon^* > 0$ such that

$$\frac{\epsilon^2}{2}(\phi'(1))^2 \ge \frac{(\beta-\alpha)^2}{K\epsilon^2} \quad \text{for } 0<\epsilon<\epsilon^*\,, \tag{3.21}$$

where $K \ge 8$ is a constant independent of ϵ. Therefore, by (3.12) and (3.21), we complete the proof of Lemma 3.3. □

3.1. Proof of Theorem 1.6. To prove Theorem 1.6, we need

Claim 3. Assume that there exists $M > 0$ independent of ϵ such that $\frac{\eta_\epsilon}{\epsilon^2} \le M$ as $0 < \epsilon << 1$. Then

$$\lim_{\epsilon\downarrow 0}\epsilon^2\left(\phi'(1)e^{\phi(1)/2}+\phi'(-1)e^{\phi(-1)/2}\right)=0. \tag{3.22}$$

Proof. By (3.13) and (3.14) we have

$$\frac{\epsilon^2}{2}\left[(\phi'(1))^2e^{\phi(1)}-(\phi'(-1))^2e^{\phi(-1)}\right]=I_+-I_- \tag{3.23}$$

where $I_\pm = \left[\frac{\alpha}{\int_{-1}^{1} e^{\phi(y)}dy}\left(e^{\phi(\pm1)}-e^{\phi(x_\epsilon)}\right)-\frac{\beta}{\int_{-1}^{1} e^{-\phi(y)}dy}e^{-\phi(x_\epsilon)}\right]e^{\phi(\pm1)}$. By (1.19), (3.15), (3.16), Lemma 3.3 and Remark 3.1 (i), we have

$$|I_\pm| \le \frac{\alpha+\beta}{2}e^{\max\{|\phi_0(1)|,|\phi_0(-1)|\}+\frac{\eta_\epsilon}{\epsilon^2}C_7}\,, \tag{3.24}$$

and

$$\epsilon^2(-\phi'(1)e^{\phi(1)/2}+\phi'(-1)e^{\phi(-1)/2}) \ge C_6\left(e^{\phi_0(1)/2}+e^{\phi_0(-1)/2}\right). \tag{3.25}$$

Then we use (3.23) and (3.25) to get

$$\epsilon^2\left|\phi'(1)e^{\phi(1)/2}+\phi'(-1)e^{\phi(-1)/2}\right| \le \frac{\epsilon^2(|I_+|+|I_-|)}{C_6\left(e^{\phi_0(1)/2}+e^{\phi_0(-1)/2}\right)}\,. \tag{3.26}$$

Therefore, by (3.24) and (3.26), we complete the proof of Claim 3. □

Suppose $\frac{\eta_\epsilon}{\epsilon^2} \to 0$ as $\epsilon \downarrow 0$. Then Lemma 3.3 and the boundary condition (1.19) imply $\lim_{\epsilon\downarrow 0} \phi(1) = \phi_0(1)$ and $\lim_{\epsilon\downarrow 0} \phi(-1) = \phi_0(-1)$. Hence by (3.22) and Remark 3.1 (ii) we obtain (1.31) and complete the proof of Theorem 1.6 (i).

Suppose $\eta_\epsilon = \gamma\epsilon^2$, where $\gamma > 0$ is a constant independent of ϵ. By Lemma 3.3 and the boundary condition (1.19), $|\phi(\pm 1)|$ has an upper bound independent of ϵ. Thus we can assume that $\limsup_{\epsilon\downarrow 0} \phi(1) = \phi_{1,s}^*$, $\liminf_{\epsilon\downarrow 0} \phi(1) = \phi_{1,i}^*$, $\limsup_{\epsilon\downarrow 0} \phi(-1) = \phi_{2,s}^*$, and $\liminf_{\epsilon\downarrow 0} \phi(-1) = \phi_{2,i}^*$. By (3.22) and the boundary condition (1.19) with $\eta_\epsilon = \gamma\epsilon^2$, we have

$$\lim_{\epsilon\downarrow 0} \left[(\phi(1) - \phi_0(1)) e^{\phi(1)/2} - (\phi(-1) - \phi_0(-1)) e^{\phi(-1)/2} \right] = 0. \tag{3.27}$$

On the other hand, (3.11) and the boundary condition (1.19) give

$$\phi(1) + \phi(-1) = \phi_0(1) + \phi_0(-1) + \gamma(\beta - \alpha). \tag{3.28}$$

By Remark 3.1 (i), (3.27) and (3.28), $(\phi_{1,s}^*, \phi_{2,s}^*)$ and $(\phi_{1,i}^*, \phi_{2,i}^*)$ solve the system (1.32). It is easy to see that system (1.32) has a unique solution. Consequently, $(\phi_{1,s}^*, \phi_{2,s}^*) = (\phi_{1,i}^*, \phi_{2,i}^*)$, and then $\lim_{\epsilon\downarrow 0} \phi(1) = \phi_1^*$, $\lim_{\epsilon\downarrow 0} \phi(-1) = \phi_2^*$, together with the boundary condition (1.19), we have $\lim_{\epsilon\downarrow 0} \epsilon^2 \phi'(1) = \frac{\phi_0(1) - \phi_1^*}{\gamma}$ and $\lim_{\epsilon\downarrow 0} \epsilon^2 \phi'(-1) = -\frac{\phi_0(-1) - \phi_2^*}{\gamma}$. Therefore, we complete the proof of Theorem 1.6 (ii).

Suppose $\frac{\eta_\epsilon}{\epsilon^2} \to \infty$ as $\epsilon \downarrow 0$. Subtracting (3.14) from (3.13) and using (3.11) and (1.19), we obtain

$$\begin{aligned} &\frac{\alpha - \beta}{2\eta_\epsilon} (\phi(-1) - \phi(1) - \phi_0(-1) + \phi_0(1)) \\ &= \frac{\alpha}{\int_{-1}^{1} e^{\phi(y)} dy} \left(e^{\phi(1)} - e^{\phi(-1)} \right) + \frac{\beta}{\int_{-1}^{1} e^{-\phi(y)} dy} e^{-\phi(-1)} \left(e^{\phi(-1) - \phi(1)} - 1 \right). \end{aligned} \tag{3.29}$$

By (3.15) and (3.16), we have

$$\left| \frac{\alpha}{\int_{-1}^{1} e^{\phi(y)} dy} \left(e^{\phi(1)} - e^{\phi(-1)} \right) \right| \leq \frac{2\alpha}{\int_{-1}^{1} e^{\phi(y)} dy} e^{\phi(x_\epsilon)} \leq \alpha + \beta \,. \tag{3.30}$$

As for the proof of Lemma 3.3, we assume $\phi'(-1) + \phi'(1) \geq 0$. Then the boundary condition (1.19) gives $\phi(-1) - \phi(1) \geq \phi_0(-1) - \phi_0(1)$. Hence by (3.6), we have

$$\left| e^{\phi(-1) - \phi(1)} - 1 \right| \geq e^{\phi_0(-1) - \phi_0(1)} \, |\phi(-1) - \phi(1)|. \tag{3.31}$$

By (3.6), (3.19), (3.29) and (3.30), we obtain $\frac{(\beta-\alpha)^2}{8\epsilon^2} \left| e^{\phi(-1)-\phi(1)} - 1 \right| \leq (\alpha + \beta) + \frac{\beta - \alpha}{2\eta_\epsilon} (\phi_0(1) - \phi_0(-1) + |\phi(-1) - \phi(1)|)$, and then by (3.31), we have

$$\begin{aligned} &\left[e^{\phi_0(-1) - \phi_0(1)} - \frac{4\epsilon^2}{(\beta - \alpha)\eta_\epsilon} \right] |\phi(-1) - \phi(1)| \\ &\leq \frac{8\epsilon^2}{(\beta - \alpha)^2} \left[(\alpha + \beta) + \frac{\beta - \alpha}{2\eta_\epsilon} (\phi_0(1) - \phi_0(-1)) \right]. \end{aligned} \tag{3.32}$$

Since $\frac{\epsilon^2}{\eta_\epsilon} \to 0$ as $\epsilon \downarrow 0$, then (3.32) gives $\lim_{\epsilon\downarrow 0} (\phi(-1) - \phi(1)) = 0$. By the boundary condition (1.19), we have $\epsilon^2(\phi'(1) + \phi'(-1)) = \frac{\epsilon^2}{\eta_\epsilon} [(\phi_0(1) - \phi_0(-1) - (\phi(1) - \phi(-1)]) \to 0$ as $\epsilon \downarrow 0$. Therefore, by (3.11), we get (1.33) and complete the proof of Theorem 1.6.

3.2. Proof of Theorem 1.5. By (3.4) and Lemma 3.3, we have $|\phi'(x)| \leq \left(\frac{\beta-\alpha}{\epsilon^2}\right)\left(e^{-\frac{\sqrt{2\alpha}}{2\epsilon}(1+x)} + e^{-\frac{\sqrt{2\alpha}}{2\epsilon}(1-x)}\right)$, for $x \in (-1,1)$. Then

$$|\phi(x)-\phi(0)| \leq \int_0^x |\phi'(s)|\, ds \leq \frac{\beta-\alpha}{\epsilon}\sqrt{\frac{2}{\alpha}}\left|e^{-\sqrt{2\alpha}(1+x)/(2\epsilon)} - e^{-\sqrt{2\alpha}(1-x)/(2\epsilon)}\right|, \quad (3.33)$$

for $x \in (-1,1)$. Hence we may complete the proof of Theorem 1.5 (i).

Now we prove Theorem 1.5(ii). Assume that $\phi_0(1) = \phi_0(-1)$. Then ϕ is even and $\phi(1) = \phi(-1)$. By (1.37) and Remark 3.1 (ii), we have

$$\frac{\alpha e^{\phi(1)}}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\beta e^{-\phi(1)}}{\int_{-1}^1 e^{-\phi(y)} dy} + \frac{\epsilon^2}{4}\int_{-1}^1 (\phi'(x))^2 dx = \frac{(\alpha-\beta)^2}{8\epsilon^2} + \frac{\alpha+\beta}{2}. \quad (3.34)$$

Setting $x = 0$ in (1.34) and integrating (1.34) from -1 to 1, we get

$$\frac{\alpha e^{\phi(0)}}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\beta e^{-\phi(0)}}{\int_{-1}^1 e^{-\phi(y)} dy} + \frac{\epsilon^2}{4}\int_{-1}^1 (\phi'(x))^2 dx = \frac{\alpha+\beta}{2}. \quad (3.35)$$

Here we have used the fact that $\phi'(0) = 0$. By Theorem 3.2 (i), $\phi(1) \leq \phi(0)$, and then (3.35) gives

$$\frac{\alpha e^{\phi(1)}}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\epsilon^2}{4}\int_{-1}^1 (\phi'(x))^2 dx \leq \frac{\alpha e^{\phi(0)}}{\int_{-1}^1 e^{\phi(y)} dy} + \frac{\epsilon^2}{4}\int_{-1}^1 (\phi'(x))^2 dx \leq \frac{\alpha+\beta}{2}. \quad (3.36)$$

Hence (3.34) and (3.36) imply

$$\frac{(\alpha-\beta)^2}{8\epsilon^2} \leq \frac{\beta e^{-\phi(1)}}{\int_{-1}^1 e^{-\phi(y)} dy} \leq \frac{(\alpha-\beta)^2}{8\epsilon^2} + \frac{\alpha+\beta}{2}. \quad (3.37)$$

By (3.35), (3.37) and (3.9), we have

$$e^{\phi(0)} \leq \frac{\alpha+\beta}{2\alpha}\int_{-1}^1 e^{\phi(y)} dy, \quad e^{-\phi(1)}\int_{-1}^1 e^{\phi(y)} dy \leq \frac{4}{\alpha}\left[\frac{(\alpha-\beta)^2}{8\epsilon^2} + \frac{\alpha+\beta}{2}\right],$$
$$e^{-\phi(0)} \leq \frac{\alpha+\beta}{2\beta}\int_{-1}^1 e^{-\phi(y)} dy, \quad e^{\phi(1)}\int_{-1}^1 e^{-\phi(y)} dy \leq \frac{8\beta\epsilon^2}{(\alpha-\beta)^2}.$$

Consequently,

$$\log\frac{(\alpha-\beta)^2}{4\epsilon^2(\alpha+\beta)} \leq \phi(0)-\phi(1) \leq \log\left\{\frac{2(\alpha+\beta)}{\alpha^2}\left[\frac{(\alpha-\beta)^2}{8\epsilon^2} + \frac{\alpha+\beta}{2}\right]\right\}. \quad (3.38)$$

Therefore, by (3.33) and (3.38), we obtain (1.27).

Now we assume $\phi_0(1) \neq \phi_0(-1)$. By Lemma 3.3, (3.17) and (3.18), we have

$$\frac{C_6^2}{2\epsilon^2} \leq \frac{\beta\, e^{-\phi(1)}}{\int_{-1}^1 e^{-\phi(y)} dy}, \quad \frac{\beta\, e^{-\phi(-1)}}{\int_{-1}^1 e^{-\phi(y)} dy} \leq \frac{C_7^2}{2\epsilon^2} + \frac{\alpha+\beta}{2}. \quad (3.39)$$

As for the proof of Theorem 1.5 (ii) for the case of $\phi_0(1) = \phi_0(-1)$, we use (3.9), (3.15) and (3.39) to derive

$$C_8 \leq \phi(x_\epsilon) - \phi(\pm 1) - \log\frac{1}{\epsilon^2} \leq C_9 \quad \text{for} \quad 0 < \epsilon < 1, \quad (3.40)$$

where C_8, C_9 are positive constants independent of ϵ. To complete the proof of Theorem 1.5 (ii), we need the following claim:

Claim 4. There exists $C_{10} > 0$ independent of ϵ, such that $0 \le \phi(x_\epsilon) - \phi(0) \le C_{10}$ as $0 < \epsilon << 1$.

Proof. Setting $x = 0$ and $x = x_\epsilon$ in (1.34), one can check that

$$\frac{\beta\, e^{-\phi(x_\epsilon)}}{\int_{-1}^{1} e^{-\phi(y)} dy}\left(e^{\phi(x_\epsilon)-\phi(0)} - 1\right) = \frac{\epsilon^2}{2}(\phi'(0))^2 + \frac{\alpha\, e^{\phi(x_\epsilon)}}{\int_{-1}^{1} e^{\phi(y)} dy}\left(1 - e^{\phi(0)-\phi(x_\epsilon)}\right). \quad (3.41)$$

By (3.4), (3.9), (3.15) and (3.41), we may complete the proof of Claim 4. ∎ Therefore, by Claim 4, (3.4) and (3.40), we get (1.27) and complete the proof of Theorem 1.5.

3.3. Proof of Theorem 1.7. By (3.9) and (3.37), we have

$$\frac{(\alpha-\beta)^2}{8\epsilon^2} e^{-\phi(x)+\phi(1)} \le \frac{\beta e^{-\phi(x)}}{\int_{-1}^{1} e^{-\phi(y)} dy} \le \left[\frac{(\alpha-\beta)^2}{8\epsilon^2} + \frac{\alpha+\beta}{2}\right] e^{-\phi(x)+\phi(1)}, \quad (3.42)$$

$$\frac{2\alpha^2\epsilon^2}{(\alpha-\beta)^2 + 4\epsilon^2(\alpha+\beta)} e^{\phi(x)-\phi(1)} \le \frac{\alpha e^{\phi(x)}}{\int_{-1}^{1} e^{\phi(y)} dy} \le \frac{2\alpha\beta\epsilon^2}{(\alpha-\beta)^2} e^{\phi(x)-\phi(1)}. \quad (3.43)$$

Let $u(x) = \phi(x) - \phi(1)$ for $x \in [-1, 1]$. Then (1.14), (3.42) and (3.43) give

$$\alpha^2 A e^{u(x)} - \frac{1}{4A} e^{-u(x)} \le \epsilon^2 u''(x) \le \alpha\beta B e^{u(x)} - \frac{1}{4B} e^{-u(x)}, \quad (3.44)$$

for $x \in (-1, 1)$ where $A = \frac{2\epsilon^2}{(\alpha-\beta)^2 + 4\epsilon^2(\alpha+\beta)}$ and $B = \frac{2\epsilon^2}{(\alpha-\beta)^2}$. Since $\phi_0(1) = \phi_0(-1)$, then by Theorem 3.2 (i), $u' \le 0$ in $(0, 1)$. Multiplying (3.44) by $u'(x)$, we find that for $x \in (0, 1)$,

$$\left(\alpha\beta B e^{u(x)} + \frac{1}{4B} e^{-u(x)}\right)' \le \epsilon^2 u''(x) u'(x) \le \left(\alpha^2 A e^{u(x)} + \frac{1}{4A} e^{-u(x)}\right)'. \quad (3.45)$$

Integrating (3.45) over $(y, 1)$ for $y \in (0, 1)$, we obtain

$$\frac{\epsilon^2}{2} u'^2(y) \le \left(\sqrt{\alpha\beta} e^{\frac{u(y)+\log B}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log B}{2}}\right)^2 + \sqrt{\alpha\beta} \quad (3.46)$$

and

$$\frac{\epsilon^2}{2} u'^2(y) \ge \alpha^2 e^{u(y)+\log A} + \frac{1}{4} e^{-(u(y)+\log A)} - \frac{\alpha+\beta}{2} - \frac{2\alpha^2\epsilon^2}{(\alpha-\beta)^2}. \quad (3.47)$$

Here we have used the fact that $u(1) = 0$ and $u'(1) = \frac{\alpha-\beta}{2\epsilon^2}$. Note that $\sqrt{\alpha\beta} e^{\frac{u(1)+\log B}{2}} - \frac{1}{2} e^{-\frac{u(1)+\log B}{2}} = \frac{1}{\sqrt{B}}\left(\frac{2\sqrt{\alpha\beta}\epsilon^2}{(\alpha-\beta)^2} - \frac{1}{2}\right)$ is negative for $0 < \epsilon < \frac{\beta-\alpha}{2(\alpha\beta)^{1/4}}$. Hence by the continuity, there exists $y_{1,\epsilon} \in (0, 1)$ sufficiently close to 1 such that

$$\sqrt{\alpha\beta} e^{\frac{u(y)+\log B}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log B}{2}} < 0 \quad \text{for } y \in (y_{1,\epsilon}, 1). \quad (3.48)$$

Consequently, by (3.46), (3.48) and Theorem 3.2(i), we have

$$0 \le -\frac{\epsilon}{\sqrt{2}} u'(y) \le -\left(\sqrt{\alpha\beta} e^{\frac{u(y)+\log B}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log B}{2}}\right) + (\alpha\beta)^{1/4}, \quad (3.49)$$

for $y \in (y_{1,\epsilon}, 1)$. On the other hand, for $0 < \epsilon < \frac{\beta-\alpha}{4\sqrt{\alpha+\beta}}$, it is easy to check that $\alpha^2 e^{u(1)+\log A} + \frac{1}{4} e^{-(u(1)+\log A)} - \frac{\alpha+\beta}{2} - \frac{2\alpha^2\epsilon^2}{(\alpha-\beta)^2} \geq -\frac{1}{2}\left(\frac{\alpha+\beta}{\alpha-\beta}\right)^2 \epsilon^2$ and $\frac{\alpha+\beta}{2} e^{\frac{u(1)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(1)+\log A}{2}} + \frac{\alpha+\beta}{\beta-\alpha}\epsilon < 0$. Hence there exists $y_{2,\epsilon} \in (0,1)$ depending on ϵ such that

$$\begin{aligned}\alpha^2 e^{u(y)+\log A} &+ \frac{1}{4} e^{-(u(y)+\log A)} - \frac{\alpha+\beta}{2} - \frac{2\alpha^2\epsilon^2}{(\alpha-\beta)^2} \\ &- \left(\frac{\alpha+\beta}{2} e^{\frac{u(y)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log A}{2}}\right)^2 \geq -\left(\frac{\alpha+\beta}{\alpha-\beta}\right)^2 \epsilon^2 \end{aligned} \tag{3.50}$$

and

$$\frac{\alpha+\beta}{2} e^{\frac{u(y)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log A}{2}} + \frac{\alpha+\beta}{\beta-\alpha}\epsilon < 0, \quad \text{for} \quad y \in (y_{2,\epsilon}, 1). \tag{3.51}$$

From (3.47), (3.50), (3.51) and Theorem 3.2(i), we have, for $y \in (y_{2,\epsilon}, 1)$,

$$0 \leq -\left(\frac{\alpha+\beta}{2} e^{\frac{u(y)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log A}{2}} + \frac{\alpha+\beta}{\beta-\alpha}\epsilon\right) \leq -\frac{\epsilon}{\sqrt{2}} u'(y). \tag{3.52}$$

Let $y_\epsilon = \max\{y_{1,\epsilon}, y_{2,\epsilon}\}$. Then by (3.49) and (3.52) we have, for $y \in (y_\epsilon, 1)$,

$$\frac{u'(y)}{\sqrt{\alpha\beta} e^{\frac{u(y)+\log B}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log B}{2}} - (\alpha\beta)^{1/4}} \leq \frac{\sqrt{2}}{\epsilon} \leq \frac{u'(y)}{\frac{\alpha+\beta}{2} e^{\frac{u(y)+\log A}{2}} - \frac{1}{2} e^{-\frac{u(y)+\log A}{2}} + \frac{\alpha+\beta}{\beta-\alpha}\epsilon}.$$

Therefore, integrating the above inequality over $(z, 1)$ for $z \in (y_\epsilon, 1)$, we may complete the proof of Theorem 1.7.

4. Comparison of PB_n and PB equations: Proof of Theorem 1.4. In this section, we compare the PB_n equation (1.14) and the PB equation (1.25) with the boundary condition (1.19). It is easy to prove that the solution $w \in C^\infty((-1,1)) \cap C^2([-1,1])$ of (1.25) and (1.19) has uniqueness which can be proved by the same method as Theorem 1.1. Besides, by (1.25), there exists constant C_ϵ such that

$$\frac{\epsilon^2}{2}(w'(x))^2 = \frac{\alpha}{2}\, e^{w(x)} + \frac{\beta}{2}\, e^{-w(x)} + C_\epsilon\,, \quad \forall x \in [-1,1], \tag{4.1}$$

and

$$\epsilon^2(w''(b)w'(b) - w''(a)w'(a)) \geq \int_a^b \left(\frac{\alpha}{2}\, e^{w(x)} + \frac{\beta}{2}\, e^{-w(x)}\right)(w'(x))^2 dx, \tag{4.2}$$

for $-1 \leq a < b \leq 1$.

As for the proof of Theorem 1.2, we can prove that w is odd, monotonically increasing on $[-1,1]$, convex on $(0,1)$ and concave on $(-1,0)$. Setting $s_\epsilon = w(1)$ and following the proof of Theorem 1.3, we obtain $\lim_{\epsilon\downarrow 0}(t_\epsilon - s_\epsilon) = 0$ and

$$\begin{aligned} 2\log &\frac{s_+(e^{s_\epsilon/2} - s_-) - s_-(e^{s_\epsilon/2} - s_+)e^{-(s_+-s_-)\sqrt{\alpha}(1-x)/2\epsilon}}{(e^{s_\epsilon/2} - s_-) - (e^{s_\epsilon/2} - s_+)e^{-(s_+-s_-)\sqrt{\alpha}(1-x)/2\epsilon}} \\ \leq & w(x) \leq 2\log\frac{(e^{s_\epsilon/2}+1) + (e^{s_\epsilon/2}-1)e^{-\sqrt{\alpha}(1-x)/\epsilon}}{(e^{s_\epsilon/2}+1) - (e^{s_\epsilon/2}-1)e^{-\sqrt{\alpha}(1-x)/\epsilon}}\,, \quad \forall\, x \in (0,1) \end{aligned} \tag{4.3}$$

where $t_\epsilon = \phi(1)$ and $s_\pm = \frac{1}{2}\left(\frac{\varepsilon\phi_0(1)}{\sqrt{\alpha}} \pm \sqrt{\left(\frac{\varepsilon\phi_0(1)}{\sqrt{\alpha}}\right)^2 + 4}\right)$. By (2.17), (4.3), and $\lim_{\epsilon\downarrow 0}(t_\epsilon - s_\epsilon) = 0$, we get (1.26) and complete the proof. Here we have used Theorem 1.2, 1.3 and the fact that both ϕ and w are odd functions on $[-1,1]$.

For $\alpha \neq \beta$ and $\phi_0(-1) = \phi_0(1)$, we obtain the following lemma:

LEMMA 4.1. *Assume that $\alpha \neq \beta$ and $\phi_0(-1) = \phi_0(1)$. Let $w \in C^\infty((-1,1)) \cap C^2([-1,1])$ be the solution of (1.25) and (1.19). Then w is an even function on $[-1,1]$ satisfying*

(i) If $\phi_0(-1) = \phi_0(1) > \frac{1}{2}\log\frac{\beta}{\alpha}$, then w is convex on $[-1,1]$ and $\phi_0(1) \geq w(x) \geq w(0) > \frac{1}{2}\log\frac{\beta}{\alpha}$ for all $x \in [-1,1]$;

(ii) If $\phi_0(-1) = \phi_0(1) < \frac{1}{2}\log\frac{\beta}{\alpha}$, then w is concave on $[-1,1]$ and $\phi_0(1) \leq w(x) \leq w(0) < \frac{1}{2}\log\frac{\beta}{\alpha}$ for all $x \in [-1,1]$.

(iii)

$$|w'(x)| \leq |w'(1)|\left(e^{-\frac{(\alpha\beta)^{1/4}}{\epsilon}(1+x)} + e^{-\frac{(\alpha\beta)^{1/4}}{\epsilon}(1-x)}\right), \quad \forall\, x \in (-1,1)\,. \tag{4.4}$$

Proof. By the uniqueness of w and $\phi_0(-1) = \phi_0(1)$, w is an even function on $[-1,1]$. Consequently, w' is odd on $[-1,1]$ and $w'(0) = 0$. By (4.2), we have

$$w''(x)w'(x) \geq 0\,, \quad \text{for } x \in (0,1)\,, \quad w''(x)w'(x) \leq 0\,, \quad \text{for } x \in (-1,0)\,. \tag{4.5}$$

Since w' is odd, (4.5) implies that w'' does not change sign on $(-1,1)$. Hence either $w'' \geq 0$ or $w'' \leq 0$ on $(-1,1)$. Suppose $w'' \leq 0$ on $(-1,1)$. Then (1.25) implies $w(x) \leq \frac{1}{2}\log\frac{\beta}{\alpha}$ for all $x \in (-1,1)$ and (4.5) implies $w'(x) \geq 0$ for $x \in (-1,0)$ and $w'(x) \leq 0$ for $x \in (0,1)$. By the boundary condition (1.19), $\phi_0(1) = w(1) + \eta_\epsilon\, w'(1) \leq \frac{1}{2}\log\frac{\beta}{\alpha}$. Hence we may get (i) by contradiction. Similarly, we prove (ii). Following the same argument as Theorem 3.2, we obtain (4.4). ☐

REMARK 4.1. *From the uniqueness of w, it is easy to prove that if $\phi_0(-1) = \phi_0(1) = \frac{1}{2}\log\frac{\beta}{\alpha}$, then there is only trivial solution $w \equiv \frac{1}{2}\log\frac{\beta}{\alpha}$.*

To see nontrivial solutions, we assume that $\phi_0(-1) = \phi_0(1) \neq \frac{1}{2}\log\frac{\beta}{\alpha}$. Then we have the following theorem:

THEOREM 4.2. *Assume $\alpha \neq \beta$ and $\phi_0(-1) = \phi_0(1) \neq \frac{1}{2}\log\frac{\beta}{\alpha}$, where α and β are positive constants independent of ϵ. Let $w \in C^\infty((-1,1)) \cap C^2([-1,1])$ be the solution of (1.25) and (1.19). Then*

(i) $\min\left\{\phi_0(1), \frac{1}{2}\log\frac{\beta}{\alpha}\right\} \leq w(x) \leq \max\left\{\phi_0(1), \frac{1}{2}\log\frac{\beta}{\alpha}\right\}$, $\forall\, x \in [-1,1]$.

(ii) $w \to \frac{1}{2}\log\frac{\beta}{\alpha}$ uniformly in any compact subset of $(-1,1)$ as ϵ goes to zero.

(iii) $\lim_{\epsilon\downarrow 0} w(1) = t$, where $t = \frac{1}{2}\log\frac{\beta}{\alpha}$ if $\lim_{\epsilon\downarrow 0}\frac{\epsilon}{\eta_\epsilon} = 0$, $t = \phi_0(1)$ if $\lim_{\epsilon\downarrow 0}\frac{\epsilon}{\eta_\epsilon} = \infty$, and $\min\left\{\phi_0(1), \frac{1}{2}\log\frac{\beta}{\alpha}\right\} < t < \max\left\{\phi_0(1), \frac{1}{2}\log\frac{\beta}{\alpha}\right\}$ is the unique zero of $|\phi_0(1) - t| = \gamma\left|\sqrt{2\alpha}e^{t/2} - \sqrt{2\beta}e^{-t/2}\right|$ if $\frac{\eta_\epsilon}{\epsilon} = \gamma$ and γ is a positive constant independent of ϵ. Moreover,

$$w_1^\epsilon(x) \leq w(x) - \frac{1}{2}\log\frac{\beta}{\alpha} \leq w_2^\epsilon(x) \quad \forall\, x \in [0,1], \tag{4.6}$$

where $w_i^\epsilon(x) = \log\dfrac{f_i^\epsilon e^{\frac{(h_i^\epsilon)^{1/4}}{\epsilon}(1-x)} + g_i^\epsilon}{f_i^\epsilon e^{\frac{(h_i^\epsilon)^{1/4}}{\epsilon}(1-x)} - g_i^\epsilon}$ for $x \in [0,1]$ and f^ϵ's, g_i^ϵ's and h_i^ϵ's are *constants satisfying* $f_i^\epsilon \to \alpha^{1/4}e^{t/2}+\beta^{1/4}$, $g_i^\epsilon \to \alpha^{1/4}e^{t/2}-\beta^{1/4}$ *and* $h_i^\epsilon \to (4\alpha\beta)^{1/4}$ *as* $\epsilon \downarrow 0^+$, $i=1,2$.

Proof. Theorem 4.2 (i) immediately follows from Lemma 4.1. Setting $x=0$ in (4.1) and using $w'(0)=0$, one may check that

$$\frac{\epsilon^2}{2}(w'(x))^2 = \alpha\left(e^{w(x)}-e^{w(0)}\right)+\beta\left(e^{-w(x)}-e^{-w(0)}\right), \forall\, x\in[-1,1]. \tag{4.7}$$

Note that $\epsilon^2 w'(1)=\epsilon^2\int_0^1 w''(x)dx=\int_0^1\left(\alpha e^{w(x)}-\beta e^{-w(x)}\right)dx$. If $w''\geq 0$ on $(0,1)$, then by (4.5), we have $w'\geq 0$ on $(0,1)$, hence $\alpha e^{w(x)}-\beta e^{-w(x)}$ is increasing on $(0,1)$, and then $\epsilon^2 w'(1)\geq \alpha e^{w(0)}-\beta e^{-w(0)}\geq 0$. Similarly, if $w''\leq 0$ on $(0,1)$, we get $\epsilon^2 w'(1)\leq \alpha e^{w(0)}-\beta e^{-w(0)}\leq 0$. Consequently,

$$\left(\alpha e^{w(0)}-\beta e^{-w(0)}\right)^2\leq \epsilon^4(w'(1))^2. \tag{4.8}$$

By Theorem 4.2 (i), (4.7) (at $x=1$) and (4.8), we have

$$\lim_{\epsilon\downarrow 0} w(0)=\frac{1}{2}\log\frac{\beta}{\alpha} \text{ and } |w'(1)|\leq \frac{C_{13}}{\epsilon}, \tag{4.9}$$

for some positive constant C_{13} independent of ϵ. Thus by (4.4) and (4.9), we complete the proof of Theorem 4.2 (ii).

By (4.7) and (1.19), we have

$$\frac{\epsilon^2}{2\eta_\epsilon^2}(\phi_0(1)-w(1))^2=\alpha\left(e^{w(1)}-e^{w(0)}\right)+\beta\left(e^{-w(1)}-e^{-w(0)}\right).$$

Using Theorem 4.2 (ii) and the proof of Theorem 1.3, we may complete the proof of Theorem 4.2 (iii). On the other hand, (4.7) can be written as

$$\frac{\epsilon^2}{2}(w'(x))^2=\left(\sqrt{\alpha}e^{w(x)/2}-\sqrt{\beta}e^{-w(x)/2}\right)^2-\left(\sqrt{\alpha}e^{w(0)/2}-\sqrt{\beta}e^{-w(0)/2}\right)^2, \tag{4.10}$$

for $x\in[-1,1]$. Note that by (1.25) and (4.5), $w'(x)$ and $\sqrt{\alpha}e^{w(x)/2}-\sqrt{\beta}e^{-w(x)/2}$ share the same sign for $x\in[0,1]$. Hence (4.10) implies

$$\frac{w'(x)}{\sqrt{\alpha}e^{w(x)/2}-\sqrt{\beta}e^{-w(x)/2}}\leq\frac{\sqrt{2}}{\epsilon}\leq\frac{w'(x)}{\sqrt{\alpha}e^{w(x)/2}-\sqrt{\beta}e^{-w(x)/2}-\delta_\epsilon},\ x\in[0,1], \tag{4.11}$$

where $\delta_\epsilon=\sqrt{\alpha}e^{w(0)/2}-\sqrt{\beta}e^{-w(0)/2}$. Note that by (4.9), we have $\delta_\epsilon\to 0$ as $\epsilon\downarrow 0$.

Therefore, as for (2.14)-(2.17), we may use (4.11) to get (4.6) and complete the proof of Theorem 4.2. ∎

5. Numerical simulations. Here we present numerical results to support the theories in this paper. For the electroneutral case i.e., $\alpha=\beta$, the Gummel method [41, 42, 43] is well-known for solving PB type equations. One may apply Gummel method to PB_n equation. However, we have empirically observed that this method shows a deviation to the solution in the nonelectroneutral case i.e., $\alpha\neq\beta$ with high electrostatic potential. It may be caused by the inaccurate correction of the exponential fitting for high electrostatic potential. Moreover, since PB_n equation has a normalization of charge density, it is not easy to find a numerical remedy in using Gummel method for the nonelectroneutral case. Due to these reasons, we here employ convex iteration [39] rather than Gummel method to linearize PB_n equation (1.14) and PB equation (1.25) with the Robin boundary condition (1.19). If ϕ_1 satisfying boundary condition (1.19) is initially given, then an iteration scheme of (1.14) can be defined with $0 < c < 1$ as follows:

$$\begin{cases} \epsilon^2 \phi''_{k+\frac{1}{2}} = \dfrac{\alpha}{\int e^{\phi_k}\,dx} e^{\phi_k} - \dfrac{\beta}{\int e^{-\phi_k}\,dx} e^{-\phi_k}, \\ \phi_{k+1} = c\phi_{k+\frac{1}{2}} + (1-c)\phi_k, \text{ for } k = 1, 2, \cdots . \end{cases} \tag{5.1}$$

with boundary conditions

$$\phi_{k+\frac{1}{2}}(-1) - \eta\phi'_{k+\frac{1}{2}}(-1) = \phi_0(-1)\,, \quad \phi_{k+\frac{1}{2}}(1) + \eta\phi'_{k+\frac{1}{2}}(1) = \phi_0(1). \tag{5.2}$$

Let $\phi_{k+\frac{1}{2}} = \phi_k + \delta_k$ with the correction δ_k which satisfies

$$\delta_k(-1) - \eta\delta'_k(-1) = 0\,, \quad \delta_k(1) + \eta\delta'_k(1) = 0\,, \tag{5.3}$$

so that $\phi_{k+1} = \phi_k + c\delta_k = \phi_1 + c\sum_{i=1}^{k}\delta_i$. If $\lim_{k\to\infty}|\delta_k| = 0$, then the iterative scheme converges.

Define the residual function $\mathcal{R}(\phi_k)$ as

$$\mathcal{R}(\phi_k) = \frac{\alpha}{\int e^{\phi_k}\,dx} e^{\phi_k} - \frac{\beta}{\int e^{-\phi_k}\,dx} e^{-\phi_k} - \epsilon^2\phi''_k, \tag{5.4}$$

then we get

$$\epsilon^2\phi''_{k+1} - \epsilon^2\phi''_k = \epsilon^2 c\delta''_k = c\mathcal{R}(\phi_k). \tag{5.5}$$

If we integrate $\mathcal{R}(\phi_{k+1}) - \mathcal{R}(\phi_k)$, then we obtain

$$\int_{-1}^{1} \mathcal{R}(\phi_{k+1})\,dx = (1-c)\int_{-1}^{1} \mathcal{R}(\phi_k)\,dx. \tag{5.6}$$

Remark 5.1. *In (5.5), it is clear that the numerical scheme converges when* $\lim_{k\to\infty}|\mathcal{R}(\phi_k)| = 0$ *under the boundary conditions (5.3). In case of* $c = 1$ *in (5.6), we can not guarantee that* $\lim_{k\to\infty}|\mathcal{R}(\phi_k)| = 0$ *but* $\int \mathcal{R}(\phi_{k+1})\,dx = 0$ *for* $k = 1, 2, \cdots$ *so that it may cause an oscillation during the iteration procedure. Moreover, we have empirically observed that when the value of* c *is compatible to* ϵ^2*, the iteration converges well.*

The finite element methods with piecewise linear basis functions are used to solve the linearized equations. The numerical computations are performed with the constants $\epsilon = 2^{-j}, j = 1, \cdots, 6$ and $\eta_\epsilon = 0, \epsilon^2, \epsilon, \epsilon^{1/2}, \epsilon^0$. The computational domain is $[-1, 1]$ and the mesh size is 1/2048 uniformly throughout all computations. The value 10^{-6} is applied for stopping criterion of iterative scheme with $|\delta_k| = |(\phi_{k+1} - \phi_k)/c|$.

5.1. Electroneutral case ($\alpha = \beta$)**.** For simplicity, we set $\alpha = \beta = 1$, and $\phi_0(1) = -\phi_0(-1) = 1$. The numerical solutions of the PB_n equation (1.14) with the Robin boundary condition (1.19) are presented in Figure 5.1. The pictures (a)-(b) of Figure 5.1 are corresponding to the case of $\epsilon = 2^{-2}, 2^{-6}$, and the curves 1-5 in each picture are associated with $\eta_\epsilon = 0, \epsilon^2, \epsilon, \epsilon^{1/2}, \epsilon^0$, respectively. Numerical outputs of $\phi(1)$'s are written at the right end of the curves 1-5 of Figure 5.1. According to the numerical results, we observe that as ϵ goes to zero, these $\phi(1)$'s close to $\phi_0(1) = 1$

when $\eta_\epsilon = 0, \epsilon^2$. $\phi(1)$'s close to 0 when $\eta_\epsilon = \epsilon^{1/2}, \epsilon^0$ as ϵ goes to zero. $\phi(1)$ close to $\phi^* = 0.4974$ when $\eta_\epsilon = \epsilon$ as ϵ goes to zero i.e., $\gamma = 1$, where $\phi^* = 0.4974$ is the root of (1.23) numerically solved by Newton's method. These results have excellent consistency with Theorem 1.2 and Theorem 1.3. One may remark Figure 5.1 to see how the solution ϕ tends to 0 in $(-1, 1)$ but may have boundary layers at $x = \pm 1$ as ϵ goes to zero.

The difference between the PB_n equation (1.14) and the PB equation (1.25) is the integrals $\int_{-1}^{1} e^{\phi(x)}dx$ and $\int_{-1}^{1} e^{-\phi(x)}dx$ located at the denominator of right hand side of (1.14). However, the numerical integrals $\int_{-1}^{1} e^{\phi(x)}dx$ and $\int_{-1}^{1} e^{-\phi(x)}dx$ are quite close to 2 as ϵ goes to zero (see the first table in Table 5.1). Such a numerical result is consistent with Lemma 2.1. Moreover, the second table in Table 5.1 shows that ϕ the solution of (1.14) approaches to w the solution of (1.25) as ϵ goes to zero. This may support Theorem 1.4.

5.2. Non-electroneutral case ($\alpha \neq \beta$)**.** We consider the case of $\alpha = 1, \beta = 2$ with $\epsilon = 2^{-2}, 2^{-4}$. Numerical solutions of the PB_n equation (1.14) are sketched in Figure 5.2 and Figure 5.3 for the case of $\phi_0(-1) = \phi_0(1) = 1$ and $\phi_0(-1) = 1, \phi_0(1) = 2$, respectively. The numerical values of $\phi(1)$ are presented on the right hand side of pictures in Figure 5.2 and Figure 5.3, and those of $\phi(-1)$ on the left hand side in Figure 5.3. As for Theorem 1.5, our numerical results may indicate the solution ϕ tends to infinity as ϵ goes to zero. Figure 5.4 is the zoom-in curve of the curve 5 in Figure 5.2 (b) to express the boundary layer of the solution ϕ with huge slopes at $x = \pm 1$ which can also be observed in Figure 5.3. These results are comparable to Theorem 1.6.

For the PB equation (1.25), numerical solutions are sketched in Figure 5.5 for the case of $\phi_0(-1) = \phi_0(1) = 1$. Such a result may support Theorem 4.2. One may compare Figure 5.2 and 5.5 to see the difference between solutions of (1.14) and (1.25).

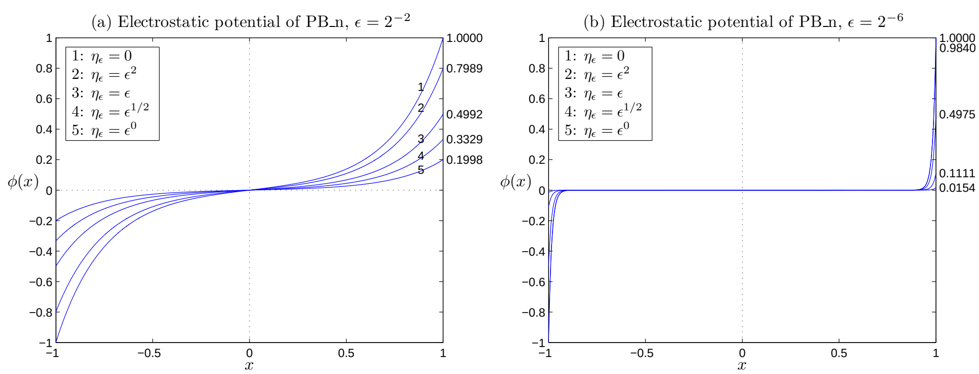

FIG. 5.1. *The numerical solutions of electrostatic potential ϕ to PB_n equation with boundary condition $\phi_0(1) = -\phi_0(-1) = 1$ in electroneutral case, $\alpha = \beta = 1$. The numerical values of $\phi(1)$ is written on the right hand side of picture for each curve. Each picture depends on the dielectric constant, (a) $\epsilon = 2^{-2}$, (b) $\epsilon = 2^{-6}$. The label of curves in the picture (b) follows that in the picture (a).*

Acknowledgements This work is done when Chiun-Chang Lee and Tai-Chia Lin were visiting IMA. They want to express sincere thanks to IMA for all the hospitality and the great working environment. Chiun-Chang Lee is supported by NCTS and NSC grant 96-2917-I-002-131 of Taiwan. The research of Tai-Chia Lin is supported

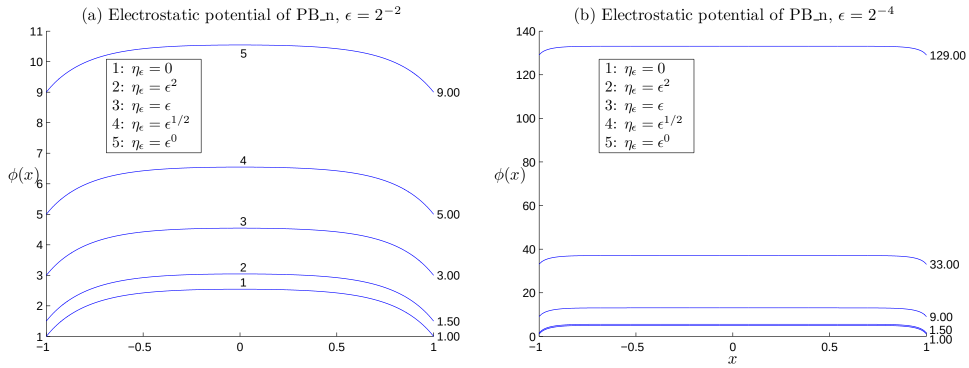

FIG. 5.2. *The numerical solutions of electrostatic potential ϕ to PB_n equation with boundary condition $\phi_0(-1) = \phi_0(1) = 1$ in non-electroneutral case, $\alpha = 1, \beta = 2$. The numerical values of $\phi(1)$ is written on the right hand side of picture for each curve. Each picture depends on the dielectric constant, (a) $\epsilon = 2^{-2}$, (b) $\epsilon = 2^{-4}$. The label of curves in the picture (b) follows that in the picture (a).*

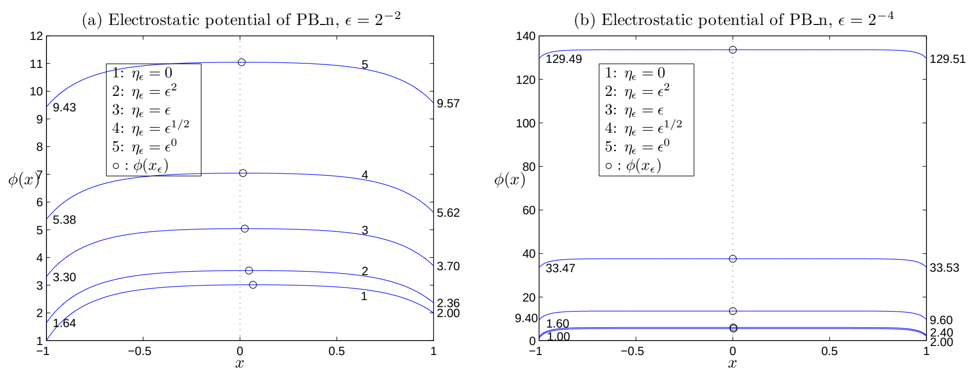

FIG. 5.3. *The numerical solutions of electrostatic potential ϕ to PB_n equation with boundary condition $\phi_0(-1) = 1, \phi_0(1) = 2$ in non-electroneutral case, $\alpha = 1, \beta = 2$. The numerical values of $\phi(1)$ is written on the right hand side of picture for each curve and those of $\phi(-1)$ on the left hand side. Each picture depends on the dielectric constant, (a) $\epsilon = 2^{-2}$, (b) $\epsilon = 2^{-4}$. The label of curves in the picture (b) follows that in the picture (a).*

by NCTS, TIMS and NSC grant 97-2918-I-002-010 of Taiwan. Chun Liu is supported by NSF grant DMS-0707594. The authors also want to thank Bob Eisenberg, Jinfang Huang, Fanghua Lin, Yoichiro Mori and Rolf Ryham for many interesting discussions and valuable comments.

6. Appendix. Here we state the proof of Theorem 1.1. Suppose $\phi_i \in C^\infty((-1,1)) \cap C^2([-1,1])$, $i = 1, 2$, are two solutions of (1.14) and (1.19). Subtracting (1.14) for ϕ_2 from that for ϕ_1, multiplying the result by $\phi_1 - \phi_2$ and then integrating the expression from -1 to 1, we obtain

$$- \epsilon^2 \int_{-1}^{1} (\phi_1 - \phi_2)''(\phi_1 - \phi_2) dx + \int_{-1}^{1} \alpha \left(e^{\phi_1 - \log \int_{-1}^{1} e^{\phi_1}} - e^{\phi_2 - \log \int_{-1}^{1} e^{\phi_2}} \right) (\phi_1 - \phi_2) dx$$

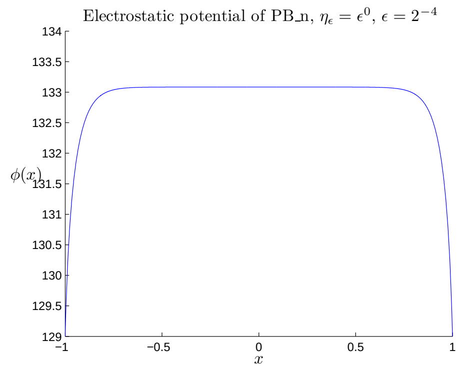

FIG. 5.4. *The zoom-in picture of the curve 5 ($\eta_\epsilon = \epsilon^0$), which is the top curve in picture (b) of Figure 5.2.*

TABLE 5.1

The numerical results of $\int_{-1}^{1} e^{\phi(x)}dx$ for PB_n equation with boundary condition $\phi_0(1) = -\phi_0(-1) = 1$ in electroneutral case, $\alpha = \beta = 1$ (first). The values of $\int_{-1}^{1} e^{-\phi(x)}dx$ are equivalent to $\int_{-1}^{1} e^{\phi(x)}dx$ in stopping criterion. The maximum norms $||\phi - w||_\infty$ of the solutions of PB_n and PB with the condition $\phi_0(1) = -\phi_0(-1) = 1$ in the case of $\alpha = \beta = 1$ (second).

$\eta_\epsilon \backslash \epsilon$	2^{-2}	2^{-4}	2^{-6}
0	2.131	2.032	2.008
ϵ^2	2.082	2.028	2.008
ϵ	2.031	2.008	2.002
$\epsilon^{1/2}$	2.014	2.001	2.000
ϵ^0	2.005	2.000	2.000

$\eta_\epsilon \backslash \epsilon$	2^{-2}	2^{-4}	2^{-6}
0	1.14e-02	2.90e-03	7.25e-04
ϵ^2	7.06e-03	2.55e-03	7.01e-04
ϵ	2.32e-03	5.83e-04	1.46e-04
$\epsilon^{1/2}$	8.14e-04	5.09e-05	2.39e-06
ϵ^0	2.02e-04	1.50e-06	7.00e-09

$$-\int_{-1}^{1} \beta \left(e^{-\phi_1 - \log \int_{-1}^{1} e^{-\phi_1}} - e^{-\phi_2 - \log \int_{-1}^{1} e^{-\phi_2}}\right)(\phi_1 - \phi_2)dx = 0. \tag{6.1}$$

Since both ϕ_1 and ϕ_2 satisfy the same boundary condition (1.19), then we have $(\phi_1 - \phi_2)(1) + \eta_\epsilon(\phi_1 - \phi_2)'(1) = (\phi_1 - \phi_2)(-1) - \eta_\epsilon(\phi_1 - \phi_2)'(-1) = 0$. Consequently, we may use integration by parts to get

$$\begin{aligned} -\epsilon^2 \int_{-1}^{1} (\phi_1 - \phi_2)''(\phi_1 - \phi_2)dx =& \epsilon^2 \int_{-1}^{1} |(\phi_1 - \phi_2)'|^2 dx \\ &+ \frac{\epsilon^2}{\eta_\epsilon}\left[(\phi_1 - \phi_2)^2(1) + (\phi_1 - \phi_2)^2(-1)\right]. \end{aligned} \tag{6.2}$$

For the last two terms of (6.1), we set $a(x) = \phi_1(x) - \log \int_{-1}^{1} e^{\phi_1}$ and $b(x) = \phi_2(x) - \log \int_{-1}^{1} e^{\phi_2}$. Note that $(e^{a(x)} - e^{b(x)})(a(x) - b(x)) \geq 0$ for $x \in [-1, 1]$ and $\int_{-1}^{1} e^{a(x)}dx = \int_{-1}^{1} e^{b(x)}dx = 1$. Hence the last two terms of (6.1) are nonnegative. Therefore, (6.1) and (6.2) imply $\phi_1 \equiv \phi_2$ and the proof of Theorem 1.1 is complete.

REFERENCES

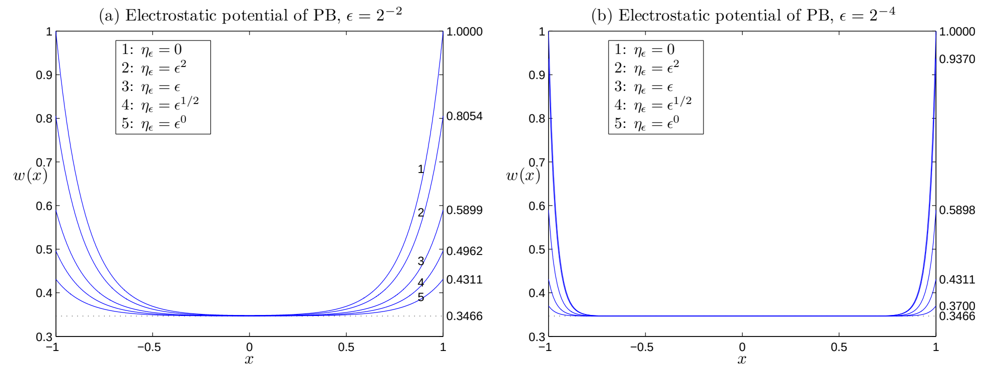

Fig. 5.5. *The numerical solutions of electrostatic potential w to PB equation with the condition $\phi_0(-1) = \phi_0(1) = 1$ for the case of $\alpha = 1$, $\beta = 2$. The numerical values of $\phi(1)$ is written on the right hand side of pictures. Each picture depends on the dielectric constant, (a) $\epsilon = 2^{-2}$, (b) $\epsilon = 2^{-4}$. The order of curves corresponding to η_ϵ follows the order in the picture (a).*

[1] V. Barcilon, D. P. Chen, and R. S. Eisenberg, Ion flow through narrow membrane channels: part II, SIAM J. APPL. MATH. Vol.52, No.5, pp.1405–1425 (1992).

[2] V. Barcilon, D. P. Chen, R. S. Eisenberg, and J. W. Jerome, Qualitative properties of steady-state Poisson-Nernst-Planck systems: perturbation and simulation study, SIAM J. APPL. MATH. Vol.57, No.3, pp.631–648 (1997).

[3] A. J Bard and L. R. Faulkner, Electrochemical Methods, John Wiley and Sons, Inc., New York, NY 2001.

[4] C. M. A. Brett and A. A. O, Brett, Electrochemistry. Principles, Methods, and Applications. Oxford Science Publications, Oxford, 1993.

[5] Kevin Taylor Chu, PhD thesis, Asymptotic Analysis of Extreme Electrochemical Transport, 2005, MIT.

[6] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics. Springer-Verlag, Berlin, (2001).

[7] P. Grochowski and J. Trylska, *Continuum molecular electrostatics, salt effects and counterion binding a review of the Poisson-Boltzmann model and its modifications*, Biopolymers 89 93-113 (2008).

[8] B. Hille, *Ion channels of excitable membranes*, 3rd Edition, Sinauer Associates, Inc. (2001).

[9] J. P. Hsu and B. T. Liu, *Current Efficiency of Ion-Selective Membranes: Effects of Local Electroneutrality and Donnan Equilibrium*, J. Phys. Chem. B 1997, 101, 7928–7932.

[10] D. Lacoste, G.I. Menon, M.Z. Bazant, and J.F. Joanny, *Electrostatic and electrokinetic contributions to the elastic moduli of a driven membrane*, Eur. Phys. J. E 28 (2009) 243–264.

[11] B. Li, *Continuum electrostatics for ionic solutions with non-uniform ionic sizes*, Nonlinearity 22 (2009) 811-833.

[12] B. Li, *Minimization of electrostatic free energy and the Poisson-Boltzmann equation for molecular solvation with implicit solvent*, SIAM J. Math. Anal. (2009) Vol.40, No. 6, pp.2536–2566.

[13] C. C. Lee, H. Lee, Y. Hyon, T. C. Lin and C. Liu, *A New Poisson-Boltzmann Type of Equations for Multi-species Ions*, preprint.

[14] C. C. Lee, T. C. Lin and C. Liu, *Steady-state solutions of the Poisson-Nernst-Planck equations with N spatial dimension, $N \geq 1$*, preprint.

[15] M. Lee and K. Y. Chan, *Non-neutrality in a charged slit pore*, Chem. Phys. Letts. 275 (1997) 56–62.

[16] W. Liu, *Geometric singular perturbation approach to steady-state Poisson-Nernst-Planck systems*, SIAM J. Appl. Math. (2005) Vol.65. No.3, pp.754–766.

[17] D.G. Luchinsky, R. Tindjong, I. Kaufman, P.V.E. McClintock and R.S. Eisenberg, *Ion channels as electrostatic amplifiers of charge fluctuations*, Journal of Physics: Conference Series 142 (2008) 012049(1-9).

[18] J. E. Marsden and A. J. Chorin, *A Mathematical Introduction To Fluid Mechanics*, Springer

(1993).
[19] J. Newman, Electrochemical Systems. Prentice-Hall Inc., Englewood Cliffs, NJ, Second edition, 1991.
[20] J. H. Park and J. W. Jerome, *Qualitative properties of steady-state Poisson-Nernst-Planck systems: mathematical study*, SIAM J. APPL. MATH. Vol.57, No.3, pp.609–630 (1997).
[21] E. Riccardi, J. C. Wang and A. I. Liapis, *Porous Polymer Absorbent Media Constructed by Molecular Dynamics Modeling and Simulations: The Immobilization of Charged Ligands and Their Effect on Pore Structure and Local Nonelectroneutrality*, J. Phys. Chem. B 2009, 113, 2317–2327.
[22] M. F. Schumaker and C. J. Kentler, *Far-Field Analysis of Coupled Bulk and Boundary Layer Diffusion Toward an Ion Channel Entrance*, Biophysical Journal Vol.74, pp.2235–2248 (1998).
[23] Y. Mori, J.W. Jerome, and C.S. Peskin, *A Three-dimensional Model of Cellular Electrical Activity*, Bulletin of the Institute of Mathematics Academia Sinica, 2(2)(2007), pp.367–390.
[24] Y. Mori, *A Three-Dimensional Model of Cellular Electrical Activity*, PhD thesis, New York University, (2007).
[25] Y. Hyon, R. Eisenberg, C. Liu, and Y. Mori, *A Mathematical Model For Electrical Activity in Cell Membrane: Energetic Variational Approach*, in preparation, (2007).
[26] A. L. Hodgkin, and A. F. Huxley, *A Qualitative Description of the Membrane Current and Its Application to Conduction and Excitation in Nerve*, J. Physiology, 117(1952), pp.500–544.
[27] J. P. Keener and J. Sneyd, *Mathematical Physiology*, Springer-Verlag, New York, 1998.
[28] C. Koch, *Biophysics of Computation*, Oxford University Press, New York, 1999.
[29] P. J. Rabier and W. C. Rheinbolt, *Theoretical and Numerical Analysis of Differential-algebraic Equations, volume VIII of Handbook of Numerical Analysis*, pp.183–540, North-Holland, 2002.
[30] S. Carl, and J. Jerome, *Trapping Region for Discontinuous Quasilinear Elliptic Systems of Mixed Monotone Type*, Nonlinear Anal., 51(2002), pp.843–863.
[31] D. J. Aidley, *The Physiology of Excitable Cells*, 4th edition, Cambridge University Press, New York, 1998.
[32] J. Zhang, X. Gong, C. Liu, W. Wen, and P. Sheng, *Electrorheological Fluid Dynamics*, PRL, 101(2008), pp.194503-1–194503-4.
[33] B. Eisenberg, *Ionic Channels in Biological Membranes: Natural Nanotubes*, Acc. Chem. Res., 31(1998), pp.117–123.
[34] W. Nonner, D. P. Chen, and B. Eisenberg, *Progress and Prospects in Permeation*, J. Gen. Physiol., 113(1999), pp.773–782.
[35] P. A. Marcowich, *The Stationary Semiconductor Device Equations*, Springer-Verlag, Vienna, 1986.
[36] O. J. Riveros, T. L. Croxton, and W. M. Armstrong, *Liquid Junction Potentials Calculated From Numerical Solutions of the Nernst-Planck and Poisson Equations*, J. Theor. Biol., 140(1989), pp.221–230.
[37] R. Ryham, *An Energetic Variational Approach To Mathematical Modeling Of Charged Fluids: Charge Phases, Simulation And Well Posedness*, thesis, Pennsylvania State University, 2006.
[38] R. Ryham, C. Liu, and L. Zikatanov, *Mathematical Models for the Deformation of Electrolyte Droplets*, Discrete Contin. Dyn. Syst. Ser. B 8 (2007), no. 3, p. 649–661.
[39] R. Ryham, C. Liu and Z.Q. Wang, *On electro-kinetic fluids: one dimensional configurations*, Discrete Contin. Dyn. Syst. Ser. B 6 (2006), no. 2, p. 357–371.
[40] Y. Hyon, D. Y. Kwak and C. Liu, *Energetic Variational Approach in Complex Fluids : Maximum Dissipation Principle*, DCDS-A, 2009, Accepted.
[41] H. K. Gummel, *A self-consistent iterative scheme for one-dimensional steady state transistor calculations*, IEEE Transactions on Electron Devices, vol. ED-11, pp. 455–465 (1964).
[42] D. L. Scharfetter and H. K. Gummel, *Large-Signal Analysis of a Silicon Read Diode Oscillator*, IEEE Transactions on Electron Devices, vol. ED-16, pp. 64–77 (1969).
[43] S. W. Bova and G. F. Carey, *Iterative Solution of the Semiconductor Device*, IEEE Transactions on Electron Devices, vol. ED-16, pp. 64–77 (1969).
[44] F. Ziebert, M. Z. Bazant and David Lacoste, *Effective zero-thickness model for a conductive membrane driven by an electric field*, Phys Rev E 81, (2010) 031912(1-13).
[45] F. Ziebert, and David Lacoste, *A Poisson.Boltzmann approach for a lipid membrane in an electric field*, New J. Phys. 12,(2010) 095002(1-15).
[46] Martin Z. Bazant, Kevin T. Chu, And B. J. Bayly, *Current-voltage relations for electrochemical thin films.*, SIAM J. APPL. MATH.,Vol. 65, No. 5, pp.1463–1484 (2005).